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LXXI. *Fitting Observations to a Curve.*

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SUMMARY.

THE simplified method of curve-fitting described by Condon, Birge, and Shea with reference to the method of least squares is applied to the method of zero sum. The relative advantages of the two methods are unchanged by the simplification of both of them. Tables are given, applicable to zero sum, similar to those given by Birge and Shea for least squares.

Some general considerations of the problem of curve-fitting are offered, suggested by the simplified method.

1. IN the University of California Publications in Mathematics (ii. pp. 55–118 (1927)) Condon, Birge, and Shea (hereafter called C.B.S.) describe a greatly simplified method of fitting observations to a curve by means of least squares. The primary object of this paper is to point out that their simplified method is equally applicable to the method of zero sum, and that the relative advantages of the two methods are unchanged when the same simplification is applied to both. But I want to take the opportunity to make

* Communicated by the Author.

some general observations on curve-fitting that are suggested by C.B.S.'s work.

The proposition on which the simplified method depends is this. We desire to fit by least squares a series of correlated observations (x, y) to a curve

$$y=\Sigma a_r x^r, \quad \ldots\ldots\ldots \quad (1)$$

where r is the set of positive integers from 0 to n—that is to say, we desire to determine the constants a_r so that the "smoothed" values of y, calculated from (1), may be used in substitution for the observed values. If we subject (1) to any linear transformation

$$x'=ax+b \quad \ldots\ldots\ldots \quad (2)$$

and fit the observations to the curve

$$y=\Sigma a_r' x'^r, \quad \ldots\ldots\ldots \quad (1.1)$$

then, although we shall, of course, arrive at different constants a_r', we shall arrive at precisely the same smoothed values of y. This proposition is useful when the x's (supposed free from all experimental error) are given at equal intervals, so that x is of the form

$$x=x_0+p\lambda, \quad \ldots\ldots\ldots \quad (3)$$

where p is the series of integers. For now the transformation (2) may be made to give (1.1) the form

$$y=\Sigma a_r' p^r \quad \ldots\ldots\ldots \quad (1.2)$$

independent of x_0 and λ. The numerical work is greatly simplified by the substitution of integers for the x's; moreover, tables can be worked out, once and for all, whereby any series of equidistant observations can be fitted to any polynomial by entering the observed y's on a simple standard form.

2. The truth of the proposition may seem to follow immediately from the fact that the linear transformation (2) corresponds in graphical smoothing to a change in zero and a change of scale; such changes ought not to affect the result of a graphical smoothing. But the matter is not quite so simple as that; for, if this were all, the proposition ought to remain true if any other form of curve were substituted for the polynomial (1). Actually it does not; it remains true only if the curve (1) is such that its form remains unchanged by the substitution, its form being determined by the number of adjustable constants a_r and their

dimensions in terms of x, y *. Thus it remains true, if the curve has the form $y = Ae^{ax}$, but not if it has the form (1), and r includes negative or fractional values. It might be suggested, therefore, that there was something less arbitrary in fitting to a curve for which the proposition is true than in fitting to one for which it is not true, so that the smoothed values depend on the choice of zero (but not on the choice of scale).

But further reflexion shows that this suggestion is false. Curve-fitting may have one of two objects: (*a*) The constants a_r may be true secondary magnitudes †; the form of the curve is then absolutely determined, and the object is to determine those constants. If the curve is not of the form for which the proposition is true, the zero will then have physical significance—choice of zero is not arbitrary at all. (*b*) The object is mere smoothing with a view to interpolation, differentiation, etc. Then any choice of form of curve is arbitrary. It is no more arbitrary to select a particular zero in fitting to a form of curve for which the proposition is true than to select one out of the many forms of curve for which it is true (*e.g.*, to select a particular degree n for the polynomial (1)). Accordingly there is no greater arbitrariness in fitting to one form of curve than to another. Smoothing must always involve some arbitrariness; there would be no such thing as smoothing if there were not certain limits within which there is nothing to decide between one arbitrary choice and another.

3. The choice between equally permissible methods of smoothing must be made on the grounds of convenience. The graphical method is usually the most convenient; it has, moreover, the advantage in principle that smoothing is an essentially graphical conception; it should therefore always be adopted if its sensitivity is sufficient ‡. Its sensitivity is most likely to be insufficient when the object of smoothing is not merely interpolation, but some analytical

* I am indebted to Mr. R. H. Fowler for pointing out to me this condition. A general proof seems not to be very easy, but there is no doubt that the condition is necessary and sufficient in all relevant instances.

† See my 'Measurement and Calculation,' chap. vi.

‡ Since the preference expressed here for a graphical method is directly contrary to that expressed in Whittaker and Robinson's standard treatise, it may be well to point out that they are concerned throughout with statistics, while I am concerned with physics. The word *accuracy* has entirely different meanings in the two sciences. To a statistician observations are accurate if the greatest possible care has been expended in obtaining them, whatever the result obtained. To a physicist (etymologically less correct) they are accurate only if

operation such as differentiation. It is only then that algebraic smoothing is desirable. Among the algebraic methods the choice of a polynomial (1) is again preferable on the score of convenience to the choice of other forms of curve, if it is permissible, and so is the choice of zero sum to least squares. It would be utterly inconsistent to adopt the polynomial rather than a curve to which the simplified C.B.S. method is inapplicable, and then to adopt a less rather than a more convenient method of smoothing to it, so long as there is no evidence that one method gives a result within the prescribed limits and the other gives a result outside them.

4. The strictest limits that can be set are defined by the rule that the residuals must be wholly irregular and that no smoothed point must diverge from an observed point by more than the maximum error determined by repetition *. However, less strict limits are often assigned; systematic residuals, exceeding the maximum error, are sometimes tolerated, especially at the ends of the curve. There may be good reason for such toleration, but once more it must be applied consistently. There is no reason to prefer one method to another, because it gives smaller residuals, unless it is clear that the larger residuals exceed some limit, assignable *à priori*, within which the smaller residuals are found to lie; if both sets of residuals are systematic, such a limit is unlikely to be found; then both methods of smoothing are usually justifiable or neither of them.

The decision whether any proposed choice of curve and any proposed method of smoothing to it are justifiable can be determined in the last resort only by an examination of the residuals; but it is one of the advantages of choosing the curves (1) that there is a simple method of deciding whether any such curve is permissible, and, if so, what must be the degree of the polynomial, which will nearly always give the right answer. It is this: take the successive diffe-

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they fulfil certain conditions, of which one is always that the observations shall be so regular that graphical smoothing does not increase the error. The observations that Whittaker and Robinson take as their example in chap. xi. are doubtless highly accurate to a statistician; but if they were presented to a physicist as the result of an experiment he would reject them as so hopelessly irregular that no important conclusion could be based on them. He would say that the experiment on which they were based did not prove what it was intended to prove, and would devise another experiment. This difference in the meaning of the word accuracy is the main reason why the computational methods worked out by statisticians are so seldom applicable to physics.

* See 'Measurement and Calculation,' chap. x.

rences (divided differences, if the x's are not equidistant) of the observations, arranged in order of x. It will often be found that the differences remain regular and decrease generally in arithmetic magnitude, as the order of the difference is increased, until the nth order is reached; thereafter they become irregular and tend to increase in arithmetic magnitude. If this is not so, no curve of the form (1) is satisfactory for smoothing. If it is so, smoothing to a polynomial of degree $(n+1)$ is likely to be satisfactory, and no improvement is likely to result from choosing a polynomial of degree greater than $(n+1)$ *.

5. We now proceed to the method of smoothing to a polynomial (1) of degree n by zero sum. Since there are $n+1$ constants to be determined, we require $n+1$ equations and $m(n+1)$ observations at equal intervals of x, where m is some integer. Let N be the number of observations; whatever is N, the observations will not usually be equidistant; whether we use least squares or zero sum, the conversion of the original observations to equidistant observations will usually be necessary. A preliminary graphical smoothing is the best plan; the sensitivity can always be made great enough by taking the differences of the observations from some very rough smoothed curve—*e.g.*, a straight line. For m the integer equal to or next greater than $N/(n+1)$ should be chosen.

The general rule of zero sum is to place in order of increasing x the observational equations that result from substituting corresponding values of x and y in (1), and to form the required $(n+1)$ equations by summing the first m, the second m, ... equations. This procedure is to be modified, in virtue of C.B.S.'s principle, only by substituting the series of integers 0, 1, 2, ... for the equidistant x's †. The resulting $(n+1)$ equations for the a_r's in (1.2) are then

* It should be observed that x and y in (1) need not be the actual observations; they may be functions of the observations, such as $\log y$ or x^2. The range of sets of observations that can be fitted to curves of the form (1) is therefore much greater than might appear at first sight. Indeed, periodic observations are the only large and important class resulting from physical experiments that cannot be fitted to this form.

† In virtue of their principle the result is precisely the same if the order of the observations is reversed, and the highest, instead of the lowest, x replaced by 0. C.B.S. work with central differences, whereas we use differences from one end. The disadvantage of central differences is that different tables are required according as N or $(n+1)$ is even or odd; the advantage is that a given accuracy can be obtained by retaining a smaller number of figures. But, as will appear presently, this advantage is less with zero sum than with least squares; with the former, therefore, differences from one end seem preferable.

$$Y_s = a_0' . S^0_{sm} + \ldots . + a_r' . S^r_{sm} + \ldots . + a_n' . S^n_{sm}, \quad (4)$$
$$(s = 0 \text{ to } n)$$

where Y_s is the sum of the $(s+1)$th set of m y's and S^r_{sm} is the sum of the rth powers of the integers from sm to $(s+1)m-1$ inclusive ($S^0_{0m} = m$). If the successive differences of these equations are taken, the coefficient of a_r' is zero in the difference of $(r+1)$th order. Accordingly, if we write

$$\left.\begin{aligned} \Delta^q Y_0 &= q\text{th order difference of } Y_0, \\ D_r{}^q = \Delta^q S^r_{0m} &= q\text{th order difference of } S^r_{0m}, \end{aligned}\right\} \quad (5)$$

we have

$$\left.\begin{aligned} Y_0 &= a_0' . S^0_{0m} + a_1' . S^1_{0m} + \ldots . + a_n' . S^n_{0m}, \\ \Delta^1 Y_0 &= a_1' . D_1' + \ldots . + a_n' . D_n', \\ &\vdots \qquad\qquad \vdots \\ \Delta^{n-1} Y_0 &= a'_{n-1} . D^{n-1}_{n-1} + a_n' . D^{n-1}_n, \\ \Delta^n Y_0 &= a_n' . D^n_n. \end{aligned}\right\} \quad (6)$$

Y_0 and $\Delta^q Y_0$ are taken from a difference table of the Y's, the coefficients S^r_{0m} and $D_r{}^q$ from tables given in the Appendix. The last equation then gives a_n'; when the value so obtained is substituted in the remainder, the last but one gives a'_{n-1}, and so on. During the solution it is convenient at the same time to insert the values of the a_r' in equations (7, 8) below.

If the problem is one of curve-fitting, the values of a_r in (1) will be required. They are obtained from

$$a_r = \Sigma_q \frac{a_q'}{\lambda^q} . (x_0)^{q-r} . {}^qC_r \ldots (q = r \text{ to } n). \quad (7)$$

If, on the other hand, the problem is one of smoothing, the smoothed values of y will be required. These are best obtained from a difference table, using the leading differences obtained from the a_r' by the equations

$$\left.\begin{aligned} y_0 &= a_0', \\ \Delta^r y_0 &= \Sigma_q \Delta^r 0^q a_q' \ldots (q = r \text{ to } n), \end{aligned}\right\} \quad (8)$$

where $\Delta^r 0^q$ is the rth order "difference of zero" of degree q. Forms for solving (7) and (8) are given in the Appendix.

The table built up from these differences will, of course, give interpolated values of y and also the derivatives $\frac{d^r y}{dp^r}$

by the usual formulæ ; from these last $\frac{d^r y}{dx^r}$ can be obtained by means of

$$\frac{d^r y}{dx^r} = \frac{1}{\lambda^r} \cdot \frac{d^r y}{dp^r} \cdot \quad \ldots \ldots \quad (9)$$

All this is the same as with C.B.S.'s method, but there is one important difference in the building-up of the table of calculated y's. If the arithmetic is correct the residuals of any group of y's summed to form a single Y must sum to zero. This proposition, characteristic of zero sum but not of least squares, is very convenient. An error of 1 in $\Delta^q y_0$ produces an error ${}^{N-1}C_q$ in the last calculated y ; accordingly, if perfect accuracy is required, many more places must be retained in $\Delta^q y_0$ than are required in y. But if too few places are retained in zero sum, the sums of the residuals corresponding to any Y will diverge from zero progressively and regularly as q increases. By examining this divergence, if it is not too large, a correction to the calculated y's may readily be found which reduces it to zero and makes the last place correct ; for this reason it is permissible to omit at least one more figure from $\Delta^q y_0$ when using zero sum than when using least squares *.

6. It remains only to apply the method to an example ; that used by C.B.S. will be taken. In Table I. the column headed y gives the observed values of y ; x increases by steps of -1 from $-1\frac{1}{2}$ for the first y to $-25\frac{1}{2}$ for the last. The columns headed $\Delta^3 y$ and $\Delta^4 y$ on the left give the third and fourth order differences of the unsmoothed y's (the first and second orders are omitted to save space). These differences are used, in accordance with the rule on p. 749, to determine the value of n in (1) that is to be adopted. It will be seen that the third order differences show some regularity, being predominantly positive, while the fourth order differences are chaotic and are tending to increase in arithmetic magnitude. Accordingly we take $n=4$, which is the value taken by C.B.S. Since the observations are equidistant in x, and since N (by a coincidence) is a multiple of $(n+1)$, we can proceed at once with the original observations, and

* It may occur to the reader that, if all these operations subsidiary to the main computation have to be performed, the "simplified" method will actually turn out to be more laborious than the direct method ; but many of them are mere substitutions for operations that are required equally in the direct method. There is no doubt whatever that, if n is as great as 3, the simplified method is a great saving.

TABLE I.

$x_0 = -1\frac{1}{2}$; $\lambda = -1$.

$\Delta^4 y$.	$\Delta^3 y$.	y.	Y.	Δ'Y.	Δ^2Y.	Δ^3.	Δ^4Y.
		3295·47					
		277·79					
	+0·27						
−0·39		257·86	16279·01				
	−0·12		(=Y_0)				
+0·14		235·95					
	+0·02						
+0·01		211·94					
	+0·03			− 651·25			
−0·01		185·85		(=$\Delta'Y_0$)			
	+0·02						
+0·03		157·71					
	+0·05						
−0·04		127·54	15627·76		− 249·61		
	+0·01				(=Δ^2Y_0)		
−0·04		095·39					
	−0·03						
+0·15		061·27					
	+0·12			− 900·86		+16·27	
−0·10		025·15				=Δ^3Y_0)	
	+0·02						
−0·11		2987·15					
	−0·09						
+0·18		947·29	14726·90		−233·34		+0·58
	+0·09						=Δ^4Y_0
−0·01		905·48					
	+0·08						
−0·11		861·83					
	−0·03			−1134·20		+16·85	
+0·09		816·42					
	+0·06						
−0·09		769·22					
	−0·03						
+0·03		720·29	13592·70		−216·49		
	±0·00						
+0·08		669·60					
	+0·08						
−0·01		617·17					
	+0·07			− 1350·69			
−0·28		563·06					
	−0·21						
+0·46		507·40					
	+0·25						
−0·34		449·98	12242·01				
	−0·09						
		391·05					
		330·52					

can omit the step of obtaining $m(n+1)$ equidistant observations. In the column headed Y the observations are

TABLE II.

$(m=5)$.

$\Delta^4 Y_0 = 75000\, a_4'$	$a_4' = +0{\cdot}000007733$
$(+0{\cdot}58)$	
$\Delta^3 Y_0 \quad -142500\, a_4' = 3750\, a_3'$	$a_3' = +0{\cdot}00404480$
$(+16{\cdot}27) \quad (-1{\cdot}102)$	
$\Delta^2 Y_0 \quad -82750\, a_4' \quad -5250\, a_3' = 250\, a_2'$	$a_2' = -1{\cdot}085940$
$(-249{\cdot}61) \quad (-0{\cdot}640) \quad (-21{\cdot}235)$	
$\Delta' Y_0 \quad -14625\, a_4' \quad -1825\, a_3' \quad -225\, a_2' = 25\, a_1'$	$a_1' = -16{\cdot}57634$
$(-651{\cdot}25) \quad (-0{\cdot}1131) \quad (-7{\cdot}3818) \quad (+244{\cdot}3365)$	
$Y_0 \quad -354\, a_4' \quad -100\, a_3' \quad -30\, a_2' \quad -10\, a_1' = 5\, a_0'$	$a_0' = +3295{\cdot}389$
$(+16279{\cdot}01) \quad (-0{\cdot}003) \quad (-0{\cdot}404) \quad (+32{\cdot}578) \quad (+165{\cdot}764)$	

$(a_r'/\lambda^r = (-1)^r a_r'\ ;\ x_0 = -1{\cdot}5,\ x_0^2 = +2{\cdot}25\ ;\ x_0^3 = -3{\cdot}375\ ;\ x_0^4 = +5{\cdot}0625)$.

$a_4 = a_4'/\lambda^4$	$= +0{\cdot}000007733$
$a_3 = -4x_0 \,.\, a_4'/\lambda^4 \quad +a_3'/\lambda^3$	$= -0{\cdot}0039984$
$(+0{\cdot}0000464) \quad (-0{\cdot}0040448)$	
$a_2 = +6x_0^2 \,.\, a_4'/\lambda^4 - 3x_0 \,.\, a_3'/\lambda^3 \quad +a_2'/\lambda^2$	$= -1{\cdot}104038$
$(+0{\cdot}000104) \quad (-0{\cdot}018202) \quad (-1{\cdot}085940)$	
$a_1 = -4x_0^3 \,.\, a_4'/\lambda^4 + 3x_0^2 \,.\, a_3'/\lambda^3 - 2x_0 \,.\, a_2'/\lambda^2 \quad +a_1'/\lambda$	$= +13{\cdot}29132$
$(+0{\cdot}00010) \quad (-0{\cdot}02730) \quad (-3{\cdot}25782) \quad (+16{\cdot}57634)$	
$a_0 = +x_0^4 \,.\, a_4'/\lambda^4 - x_0^3 \,.\, a_3'/\lambda^3 + x_0^2 \,.\, a_2'/\lambda^2 - x_0 \,.\, a_1'/\lambda \quad +a_0'$	$= +3317{\cdot}797$
$(0\ 000) \quad (-0{\cdot}0137) \quad (-2{\cdot}443) \quad (+24{\cdot}865) \quad (+3295{\cdot}389)$	

$\lambda^4 y_0 = 24\, a_4'$	$= +0{\cdot}0001856$
$\Delta^3 y_0 \quad = 36\, a_4' \quad +6\, a_3'$	$= +0{\cdot}024547$
$(+{\cdot}0002784) \quad (+{\cdot}0242688)$	
$\Delta^2 y_0 \quad = 14\, a_4' \quad +6\, a_3' \quad +2\, a_2'$	$= -2{\cdot}14750$
$(+{\cdot}000108) \quad (+0{\cdot}024269) \quad (-2{\cdot}171880)$	
$\Delta y_0 \quad = a_4' \quad +a_3' \quad +a_2' \quad +a_1'$	$= -17{\cdot}6583$
$(+{\cdot}00001) \quad (+{\cdot}00404) \quad (-1{\cdot}08594) \quad (-16{\cdot}57634)$	

summed in sets of five; in the remaining columns on the right these sums are differenced and the values of $\Delta^r Y_0$ obtained.

In Table II. the equations (6), (7), and (8) are set up by means of the tables in the Appendix and solved. No further

explanation of the procedure should be necessary, but it will be observed that the order of the equations has been reversed for convenience.

In Table III. the smoothed or calculated y's are obtained from a difference table, using the values of $\Delta^r y_0$ found in Table II. To save space only the first part of the Table is given; it is to be observed that the residuals for each of the first two sets of five y's add to zero, as they should.

Finally, the accompanying figure shows the residuals obtained by zero sum and by least squares. In the first few values of x the residuals show an obvious trend (there is some

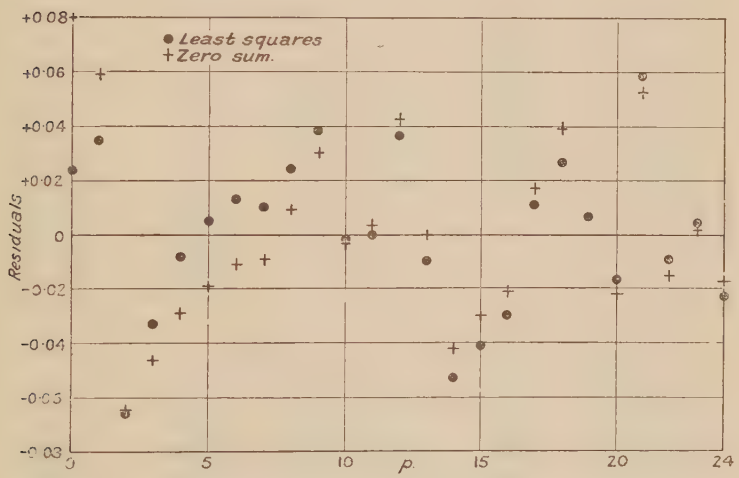
Table III.

$\Delta^4 y$.	$\Delta^3 y$.	$\Delta^2 y$.	$\Delta^1 y$.	y (calc.).	y (obs.)$-y$ (calc.).	
				3295·389	+0·081	
			−17·6583			
		−2·14750		277·731	+0·059	
	+0·024547		−19·8058			sum
+·0001856		−2·12295		257·925	−0·065	=0·000
	+0·024733		−21·9287			
		−2·09822		235·996	−0·046	
	+0·024918		−24·0269			
		−2·07330		211·969	−0·029	
	+0·025104		−26·1002			
		−2·04819		185·869	−0·019	
	+0·025290		−28·1484			
		−2·02290		157·721	−0·011	
	+0·025475		−30·1713			sum
		−1·99743		127·549	−0·009	=0·000
	+0·025661		−32·1687			
		−1·97177		095·381	+0·009	
			−34·1405			
				061·240	+0·030	

sign of a repetition of the same trend for later values of x); neither method of smoothing satisfies the strict criterion of p. 748, both import something that is not justified by the observations themselves. Further, the trend of the residuals cannot be removed by adopting a larger value of n, unless n becomes comparable with N and there is no smoothing. Whether, in these circumstances, smoothing to a polynomial is justified at all is a question that cannot be answered without a thorough inquiry into the meaning and origin of the observations; but it is most unlikely that such an inquiry would show that the difference between the two methods of smoothing (which is marked only in the first two values of x) is such that one is justifiable and the other is not.

As usual, it turns out that, when there is any justification for adjusting observations by either least squares or by zero sum, the two methods yield physically indistinguishable results.

Once more, then, I would plead with the adherents of least squares. Physicists generally do not study computational methods deeply; when they are forced to use them, they accept what they are told by those who have made them their special study. Such persons are accepting a very heavy responsibility if they induce their colleagues to waste unnecessary labour without offering any evidence that the traditional methods are superior.



Appendix.

Table IV. gives the coefficients of a_r' in equations (6) sufficient to fit to any polynomial of degree not greater than 6. A different set of coefficients is required for each value of m, but the coefficient of any a_r' is the same whatever the value of n. Accordingly the values for equations of lower degree than the sixth are obtained by simply omitting the higher terms (see example, where the terms a_5' and a_6' are simply omitted from the table for $m=5$). Values for $m > 10$ are not likely to be required, for accuracy is very seldom gained by having more than ten observations in each group.

The coefficients can be calculated either by forming the

sums S^r_{sm} and differencing them, or, more conveniently, by using the formula

$$\Delta^q S^r_{0m} = \Sigma_t {}^rC_t . S^t_{0m} . m^{r-t} . \Delta^q 0^{r-t} \quad (t=0 \text{ to } r-q).$$

TABLE IV.

$m=2.$	$a_0'.$	$a_1'.$	$a_2'.$	$a_3'.$	$a_4'.$	$a_5'.$	$a_6'.$
$Y_0 =$	2	1	1	1	1	1	1
$\Delta^1 Y_0 =$		4	12	34	96	274	792
$\Delta^2 Y_0 =$			16	120	688	3600	18,136
$\Delta^3 Y_0 =$				96	1344	12,960	107,520
$\Delta^4 Y_0 =$					768	17,280	251,520
$\Delta^5 Y_0 =$						7680	253,440
$\Delta^6 Y_0 =$							92,160

$m=3.$	$a_0'.$	$a_1'.$	$a_2'.$	$a_3'.$	$a_4'.$	$a_5'.$	$a_6'.$
$Y_0 =$	3	3	5	9	17	33	65
$\Delta^1 Y_0 =$		9	45	207	945	4359	20,385
$\Delta^2 Y_0 =$			54	648	5886	48,600	385,614
$\Delta^3 Y_0 =$				486	10,692	161,190	2,084,940
$\Delta^4 Y_0 =$					5832	204,120	4,607,280
$\Delta^5 Y_0 =$						87,480	4,461,480
$\Delta^6 Y_0 =$							1,574,640

$m=4.$	$a_0'.$	$a_1'.$	$a_2'.$	$a_3'.$	$a_4'.$	$a_5'.$	$a_6'.$
$Y_0 =$	4	6	14	36	98	276	794
$\Delta^1 Y_0 =$		16	112	712	4480	28,456	183,232
$\Delta^2 Y_0 =$			12	2112	26,240	295,680	3,197,888
$\Delta^3 Y_0 =$				1536	46,080	944,640	16,588,800
$\Delta^4 Y_0 =$					24,576	1,167,360	35,696,640
$\Delta^5 Y_0 =$						491,520	33,914,880
$\Delta^6 Y_0 =$							11,796,480

$m=5.$	$a_0'.$	$a_1'.$	$a_2'.$	$a_3'.$	$a_4'.$	$a_5'.$	$a_6'.$
$Y_0 =$	5	10	30	100	354	1300	4890
$\Delta^1 Y_0 =$		25	225	1825	14,625	118,225	968,625
$\Delta^2 Y_0 =$			250	5250	82,750	1,181,250	16,171,750
$\Delta^3 Y_0 =$				3750	142,500	3,693,750	81,937,500
$\Delta^4 Y_0 =$					75,000	4,500,000	173,625,000
$\Delta^5 Y_0 =$						1,875,000	163,125,000
$\Delta^6 Y_0 =$							56,250,000

TABLE IV. (*cont.*).

$m=6$.	a_0'.	a_1'.	a_2'.	a_3'.	a_4'.	a_5'.	a_6'.
$Y_0 =$	6	15	55	225	979	4425	20,515
$\Delta^1 Y_0 =$		36	396	3906	38,016	373,026	3,708,936
$\Delta^2 Y_0 =$			432	11,016	210,384	3,635,280	60,209,352
$\Delta^3 Y_0 =$				7776	357,696	11,210,400	300,464,640
$\Delta^4 Y_0 =$					186,624	13,530,240	630,322,560
$\Delta^5 Y_0 =$						5,598,720	587,865,600
$\Delta^6 Y_0 =$							201,553,920

$m=7$.	a_0'.	a_1'.	a_2'.	a_3'.	a_4'.	a_5'.	a_6'.
$Y_0 =$	7	21	91	441	2275	12,201	67,171
$\Delta^1 Y_0 =$		49	637	7399	84,721	977,599	11,428,417
$\Delta^2 Y_0 =$			686	20,580	461,678	9,363,900	181,969,046
$\Delta^3 Y_0 =$				14,406	777,924	28,595,910	898,502,220
$\Delta^4 Y_0 =$					403,368	34,286,280	1,871,627,520
$\Delta^5 Y_0 =$						14,117,880	1,736,499,240
$\Delta^6 Y_0 =$							592,950,960

$m=8$.	a_0'.	a_1'.	a_2'.	a_3'.	a_4'.	a_5'.	a_6'.
$Y_0 =$	8	28	140	784	4676	29,008	184,820
$\Delta^1 Y_0 =$		64	960	12,832	168,960	2,241,184	30,113,280
$\Delta^2 Y_0 =$			1024	35,328	910,336	21,196,800	472,743,424
$\Delta^3 Y_0 =$				24,576	1,523,712	64,266,240	2,316,042,240
$\Delta^4 Y_0 =$					786,432	76,677,120	4,799,201,280
$\Delta^5 Y_0 =$						31,457,280	4,435,476,480
$\Delta^6 Y_0 =$							1,509,949,440

$m=9$.	a_0'.	a_1'.	a_2'.	a_3'.	a_4'.	a_5'.	a_6'.
$Y_0 =$	9	36	204	1296	8772	61,776	446,964
$\Delta^1 Y_0 =$		81	1377	20,817	309,825	4,644,081	70,503,777
$\Delta^2 Y_0 =$			1458	56,862	1,654,830	43,499,430	1,094,944,878
$\Delta^3 Y_0 =$				39,366	2,755,620	131,154,390	5,332,124,700
$\Delta^4 Y_0 =$					1,417,176	155,889,360	11,004,371,640
$\Delta^5 Y_0 =$						63,772,920	10,139,894,280
$\Delta^6 Y_0 =$							3,443,737,680

TABLE IV. (*cont.*).

$m=10$.	a_0'.	a_1'.	a_2'.	a_3'.	a_4'.	a_5'.	a_6'.
$Y_0 =$	10	45	285	2025	15,333	120,825	978,405
$\Delta^1 Y_0 =$		100	1900	32,050	532,000	8,891,650	150,499,000
$\Delta^2 Y_0 =$			2000	87,000	2,822,000	82,650,000	2,317,499,000
$\Delta^3 Y_0 =$				60,000	4,680,000	248,100,000	11,232,000,000
$\Delta^4 Y_0 =$					2,400,000	294,000,000	23,106,000,000
$\Delta^5 Y_0 =$						120,000,000	21,240,000,000
$\Delta^6 Y_0 =$							7,200,000,000

Table V. gives the coefficients of a_q/λ^q in the equation for a_r according to (7). Table VI. gives the coefficients of a_q in the equation for $\Delta^r y_0$ according to (8).

TABLE V.

	a_0'.	a_1'/λ.	a_2'/λ^2.	a_3'/λ^3.	a_4'/λ^4.	a_5'/λ^5.	a_6'/λ^6.
$a_0 = 1$		$-x_0$	$+x_0^2$	$-x_0^3$	$+x_0^4$	$-x_0^5$	$+x_0^6$
$a_1 =$		1	$-2x_0$	$+3x_0^2$	$-4x_0^3$	$+5x_0^4$	$-6x_0^5$
$a_2 =$			1	$-3x_0$	$+6x_0^2$	$-10x_0^3$	$+15x_0^4$
$a_3 =$				1	$-4x_0$	$+10x_0^2$	$-20x_0^3$
$a_4 =$					1	$-5x_0$	$+15x_0^2$
$a_5 =$						1	$-6x_0$
$a_6 =$							1

TABLE VI.

	a_0'.	a_1'.	a_2'.	a_3'.	a_4'.	a_5'.	a_6'.
$y_0 =$	1	0	0	0	0	0	0
$\Delta^1 y_0 =$		1	1	1	1	1	1
$\Delta^2 y_0 =$			2	6	14	30	62
$\Delta^3 y_0 =$				6	36	150	540
$\Delta^4 y_0 =$					24	240	1560
$\Delta^5 y_0 =$						120	1800
$\Delta^6 y_0 =$							720

LXXII. *The Magneto-Optical Dispersion of Organic Liquids.* Part I. *The Magneto-Optical Dispersion of Normal Butyl Alcohol, Isobutyl Alcohol, and Propionic Acid.* *By* Emlyn Stephens, *M.Sc., and* Prof. E. J. Evans, *D.Sc., Physics Department, University College of Swansea* *.

Introduction.

A STUDY of the magneto-optical rotations of organic liquids provides valuable information concerning the nature and properties of molecules. The problem has been investigated from this point of view by Perkin †, who determined the molecular rotations for sodium light of a large number of organic liquids belonging to various homologous series. The rotation, however, depends upon the wave-length, and although Lowry ‡ and others have examined the magneto-optical dispersion of many organic liquids, our information on this aspect of the problem is still very incomplete. The present investigation is a continuation of previous work carried out by Stephens and Evans § etc. on the magneto-optical dispersion of organic liquids in the ultra-violet region of the spectrum. The problem of magneto-optical rotation has been considered from the theoretical side by Larmor ‖, Richardson ¶, and others, and the experimental results of the present investigation have been considered in relation to their work.

According to Larmor the magneto-optical rotation is a kinetic phenomenon related to the free periods of the resonators in the molecule. If the natural dispersion of a substance is known, it is possible from the measurements of its magneto-optical dispersion to deduce important information with regard to the nature and number of resonators present in the molecule, together with their free periods.

For regions of the spectrum not too near an absorption band the natural dispersion of a substance can be represented accurately by the following equation of the Ketteler-Helmholtz type :—

$$n^2-1=b_0+\frac{b_1}{\lambda^2-\lambda_1{}^2}+\frac{b_2}{\lambda^2-\lambda_2{}^2}+\ldots, \quad . \quad . \quad (1)$$

* Communicated by the Authors.

† Journ. Chem. Soc. p. 468 (1884).

‡ Journ. Chem. Soc. p. 1067 (1913).

§ Phil. Mag. iii. (March 1927): Jones & Evans, Phil. Mag. v. (March 1928); Evans & Evans, Phil. Mag. viii. (August 1929).

‖ 'Æther and Matter,' Appendix F, p. 352.

¶ Phil. Mag. xxxi. pp. 232 & 454.

where n is the refractive index for wave-length λ; b_0, b_1, b_2, &c., are constants ; and λ_1, λ_2 .. are the wave-lengths of the absorption bands of the substance.

According to Larmor the value of Verdet's constant, δ, is given by the equation

$$\delta = \frac{e}{2mC^2}\lambda\frac{dn}{d\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where C is the velocity of light, and $\frac{e}{m}$ is the ratio of charge to mass for all the resonators. In deducing this equation the charge e is measured in electrostatic units and the field in electromagnetic units.

If the value of $\frac{dn}{d\lambda}$ obtained from equation (1) is now put into equation (2), it can be shown * that

$$\phi = n\delta\lambda^2 = K_1\left\{\frac{\lambda^2}{\lambda^2-\lambda_1^2}\right\}^2 + K_2\left\{\frac{\lambda^2}{\lambda^2-\lambda_2^2}\right\}^2 + \ldots, \quad (3)$$

where K_1, K_2, &c. are constants, and λ_1, λ_2, &c. are wave-lengths corresponding to the free periods of the resonators.

For substances which are transparent in the visible and infra-red, λ_1, λ_2, &c., represent wave-lengths corresponding to ultra-violet free periods, and when a substance possesses only one absorption band in the ultra-violet contributing to the magnetic rotation, the equation takes the form

$$\phi = K_1\left\{\frac{\lambda^2}{\lambda^2-\lambda_1^2}\right\}^2 . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From the two values of ϕ corresponding to two values of λ it is possible to determine K_1 and λ_1 for the region of the spectrum between the two wave-lengths chosen. If the experimental results lead to different values of λ_1, depending on the wave-lengths chosen, and if the values change progressively when the chosen wave-lengths are taken in regions of shorter and shorter wave-lengths, it is presumed that the substance has more than one free period. If, however, the values of λ_1 are constant within experimental error, the values of ϕ are given by equation (4).

Experimental Arrangement.

A parallel beam of light was obtained by placing a tungsten arc at the focus of a quartz lens. This lens, together with the modified Jellett prism † immediately behind it, formed

* Richardson, *loc. cit.*

† 'Dictionary of Applied Physics,' iv p. 476.

the polarizing unit of the Bellingham-Stanley polarimeter. This polarimeter had been specially designed for work in the ultra-violet region of the spectrum, as the components of the polarizer were assembled without any intervening film which may cause absorption in that region. The beam of light, after emerging from the polarizer, consisted of two semicircular beams, the vibrations in one half being polarized at a slight angle with those in the other, the line of demarcation between them being horizontal. The beam then traversed the liquid contained in a tube which was fixed in position inside a solenoid. On emerging from the liquid the beam falls on the analyser and is brought to a focus on the slit of a quartz spectrograph by a quartz-fluorite lens, which formed part of the Bellingham-Stanley analyser unit. The analyser was of the usual type, its component parts being assembled with a thin film of glycerine. Both the analyser and the quartz-fluorite lens behind it were mounted in the same brass tube, and the lens could be independently adjusted so that the beam of light passed axially through the apparatus. The analyser unit, which could be rotated, had a vernier scale fixed to it, and the rotation could be measured on a fixed circular scale to 1′ of arc.

The same polarimeter tube was used throughout the investigation. It was made of clear fused quartz, having an external diameter of about 1 cm., and could contain a column of liquid 30·5 cm. long. The two ends of the tube were closed by fusing on disks of polished fused quartz. The disks were fused along the outside circumference of the tube, and there was no distortion of light by the central portions of the disks. These disks rotate polarized light in a magnetic field, so that a correction has to be applied when measuring magneto-optical rotations of liquids. The value of the actual correction to be applied at any wave-length could be read off from a curve obtained by the experimental determination of the rotations of the disks corresponding to several wave-lengths.

The solenoid, which was wound on a brass tube 47·3 cm. long and 4·27 cm. diameter, was connected in series with a battery of accumulators, two variable resistances, a Weston ammeter and a standard ammeter made by the Cambridge Instrument Company. One of these resistances was fitted with a fine adjustment, and the current of 2 amperes employed could be kept constant to within 1 part in 1000. One ammeter served as a check on the other, so that any change in the behaviour of one of the instruments during

the progress of the work could be easily detected. The ammeters were also periodically calibrated by comparison with a large standard Weston ammeter. By passing water through the jacket between the coil and the polarimeter tube the temperature of the liquid in the tube could be kept constant to within 0·2° C. for long intervals of time, and the variation of temperature during an exposure of only fifteen minutes was less than 0·1° C.

The magnetic field * at various points inside the solenoid had been previously measured by means of a delicate ballistic galvanometer and an accurately constructed search-coil, and the value of ΣHl for the polarimeter tube found to be 12,270 cm. gauss when a current of 2 amperes passed through the solenoid. In the expression ΣHl, l represents the length of liquid over which H can be assumed to be constant. In previous experiments Stephens and Evans † had determined the values of Verdet's constant for conductivity-water at different wave-lengths. During the present investigation experiments on the rotation of conductivity-water at different wave-lengths were periodically carried out in order to check the value obtained for ΣHl when a current of 2 amperes passed through the solenoid. In all cases the value was found to be 12,270 cm. gauss within experimental error.

The spectrograph employed was of the usual type, and consisted of a Cornu quartz prism, together with quartz lenses. The glycerine films employed in assembling the components of the analyser and the Cornu prism absorbed practically no light in the region of the spectrum investigated, since with the three liquids examined it was found impossible to measure Verdet's constant for wave-lengths below ·32 μ, even with such a powerful source of ultra-violet radiation as the tungsten arc.

Wellington special rapid plates were used, and the exposures, two of which were taken on each plate, varied from about 4 minutes at ·43 μ to about 30 minutes at ·32 μ.

Mode of Experiment.

The two beams of plane polarized light, after emerging from the polarimeter, appear at the camera end of the spectroscope as two spectra, one on top of the other, separated by a fine horizontal gap. These spectra are in general of different intensity, but in the absence of the

* Evans & Evans, *loc. cit.*

† *Loc. cit.*

magnetic field, and with no liquid in the tube, it is always possible, by rotating the analyser, to make the two spectra have the same intensity throughout their length. They are only equally intense for four definite positions of the analyser, with an angle of 90° separating one position from the next. Two of these positions, separated from one another by 180°, correspond to the maximum intensity, and the other two to the minimum intensity. In this latter position the instrument is very sensitive, and one of these positions is used as the zero reading of the analyser.

The position of the zero could be determined within a few minutes of arc by direct observation in the visible, but a series of photographs corresponding to different settings of the analyser near the position of minimum intensity were necessary for determining the exact position. With the liquid in the tube and the magnetic field not excited a few photographs near the zero position would be sufficient to show whether the liquids contained a small percentage of an optically active impurity. The presence of a very small percentage can be detected, as the column of liquid is 30·5 cm. long, and the rotation can be examined in the ultra-violet region of the spectrum.

With these preliminaries over, the determination of the magnetic rotation could be commenced. The magnetic field was applied, and a stream of water allowed to flow through the jacket. It generally required an interval of about one hour to adjust the flow of water so that the polarimeter tube was at the required temperature and for the conditions to become steady. When the temperature of the liquid in the tube was not steady the image of the two semicircular half-fields produced by the quartz-fluorite lens on the slit of the spectroscope was distorted. The analyser was then rotated through a known angle θ_1 from the zero, and a photograph taken. It is important to note that in measuring angles of rotation the readings on both verniers were taken. During the exposure a constant watch was kept over the reading of the ammeters, the temperature of the water, and the position of the arc light. With the magnetic field reversed, and a rotation of θ_1 on the opposite side of the zero, a second photograph was then taken on the same plate.

An examination of the plate shows that whilst the upper and lower halves of each spectrum are of very different intensities, there is a line of definite wave-length which has the same intensity in both halves. When the zero position had been accurately determined the wave-length of this line

is very nearly the same for each direction of the magnetic field. The reversal of the magnetic field supplies a means of eliminating certain sources of error, such as a slight mistake in the location of the zero position and an error in the centring of the circular scale of the analyser.

If λ is the mean wave-length of the lines which give equality of intensity in the two spectra, and θ_2 the magnetic rotation of the quartz disks at wave-length λ, then the rotation produced by the liquid at wave-length λ is $(\theta_1-\theta_2)$. Then, Verdet's constant δ is given by

$$\delta.\Sigma Hl=\theta=(\theta_1-\theta_2)=\delta\times 12{,}270.$$

A series of determinations of δ were made for each liquid, corresponding to increasing values of θ, until the point of equality had shifted to the limit in the ultra-violet.

The accuracy of the experimental results is chiefly affected by the difficulties involved in measuring the value of ΣHl for the tube, and in estimating the wave-lengths of the lines of equal intensities on the photographic plate. The measurement of ΣHl had been previously carried out by Evans and Evans *, who found that the experimental value agreed with that calculated from the dimensions of the coil to within one in a thousand. Taking all factors into consideration it is considered that the experimental results are correct to within one part in three hundred on the average. This conclusion is supported by the fact that the values of λ for sodium light deduced from the equation connecting δ with λ are in good agreement with those obtained experimentally by Perkin †. It is, however, important to remember that the positions of the absorption bands as calculated from the measured rotations for two wave-lengths are not affected by an inaccuracy in the determination of the absolute value of ΣHl for the tube.

The three liquids employed in this work were the purest obtainable from Dr. Schuchardt, of Görlitz, and the specimen of butyl alcohol was marked exceptionally pure. However, each liquid was subjected to a process of fractional distillation before using, and the fraction retained which came over at the boiling point given in the most recent literature. The process of fractional distillation showed that propionic acid and isobutyl alcohol were reasonably pure, and the special purity of the butyl alcohol was indicated by the fact that all the liquid distilled over at a constant temperature.

* *Loc. cit.* † *Loc. cit.*

The Measurement of Natural Dispersion.

As one of the main objects of the present investigation was to test the application of equation (4) and calculate the positions of the absorption bands, it was necessary to obtain the values of the refractive indices of the three liquids for wave-lengths in the visible and ultra-violet. These values could not, however, be obtained from existing tables, and consequently the ordinary dispersion of the three liquids was investigated.

The same spectrograph was used in these determinations as for the magneto-optical dispersion, but the Cornu prism was replaced by a hollow prism closed by plates of polished fused quartz. The prism was first filled with a standard liquid whose refractive indices are known over a wide range of wave-lengths. Light from a copper arc, after passing through a quartz lens, was brought to a focus on the slit, immediately in front of which was supported a metal plate perforated with three holes slightly overlapping. A photograph was taken with the top portion of the slit illuminated. The liquid was then completely removed from the prism by evaporation caused by an air current, care being taken not to shift the prism, which had been rigidly fixed in position. The prism was then filled with the liquid whose dispersion was required, and a second photograph was taken on the same plate with the middle portion of the slit illuminated. With the standard liquid again placed in the prism, a third photograph was taken on the same plate with the bottom portion of the slit illuminated. In this way three spectra slightly overlapping were obtained on the photographic plate, and by observation of the top and bottom spectra any displacement of the prism or photographic plate, and any displacement of the spectra due to slight changes of temperature, could be observed. The temperature was observed on a thermometer placed inside the spectrograph near the prism, and an interval of time was allowed to elapse between successive photographs in order that the liquid should attain the temperature of the air inside the spectrograph.

To find the refractive indices at various wave-lengths for the substances examined, it is only necessary to identify a line of wave-length λ_1, say, in the spectrum due to refraction through the standard liquid which is superposed upon a line of wave-length λ_2 due to refraction through the substance. The refractive index of the substance for wave-length λ_2 is then equal to the refractive index of the

standard liquid for wave-length λ_1. The standard liquid employed in these experiments was propyl alcohol, and the values of its refractive indices over a wide range of wave-lengths have been accurately measured by Victor Henri *. The chief difficulty encountered in accurate determinations of refractive indices is the variation in the values caused by small changes of temperature of one or two degrees, and the small shifts found on some of the plates were in all probability due to this cause. It is claimed, however, that on the average the refractive indices are correct to within

TABLE I. (*a*).

Normal Butyl Alcohol.

Spectrometer Determinations.		Photographic Determinations.	
Wave-length (microns).	Refractive index.	Wave-length (microns).	Refractive index.
·6678	$1·400_2$	·5105	$1·405_8$
·5876	$1·402_4$	·4545	$1·409_2$
·5016	$1·406_2$	·4375	$1·410_3$
·4922	$1·406_8$	·4275	$1·411_6$
·4713	$1·408_2$	·4022	$1·414_5$
·4472	$1·409_0$	·3861	$1·416_4$
		·3688	$1·418_6$
		·3600	$1·420_0$
		·3454	$1·423_0$
		·3339	$1·425_5$
		·3247	$1·427_5$
		·2997	$1·434_2$
		·2805	$1·441_2$

Temperature..... 13·0° C.

one part in 1400, and this degree of accuracy is more than sufficient for evaluating the wave-lengths of the absorption bands from the magneto-optical dispersion experiments. The refractive indices of the three liquids in the region of the visible spectrum, extending from ·6678 μ to ·4472 μ, were determined accurately at a known temperature by means of a spectrometer with a circular scale reading to 10″. The slit of the spectrometer was illuminated by the capillary of a helium spectrum tube, and the angles of minimum deviation

* 'Études de Photochemie,' p. 61.

were measured for the strong helium lines when the prism was filled with each liquid in turn. The values of the refractive indices obtained by the photographic method overlapped those obtained by the visual method over a certain region of the spectrum, and the effect of slight shifts visible on some of the plates could thus be adjusted.

Experimental Results.

Natural Dispersion.

The refractive indices of normal butyl alcohol, isobutyl alcohol, and propionic acid in the visible and ultra-violet

Table I. (*b*).

Isobutyl Alcohol.

Spectrometer Determinations.		Photographic Determinations.	
Wave-length (microns).	Refractive index.	Wave-length (microns).	Refractive index.
·6678	$1{\cdot}396_3$	·4545	$1{\cdot}405_4$
·5876	$1{\cdot}398_6$	·4375	$1{\cdot}406_6$
·5016	$1{\cdot}402_4$	·4275	$1{\cdot}407_7$
·4922	$1{\cdot}403_0$	·4022	$1{\cdot}410_4$
·4713	$1{\cdot}404_2$	·3860	$1{\cdot}412_5$
·4472	$1{\cdot}406_1$	·3655	$1{\cdot}415_5$
		·3530	$1{\cdot}417_4$
		·3415	$1{\cdot}419_5$
		·3337	$1{\cdot}421_1$
		·3247	$1{\cdot}423_5$
		·2961	$1{\cdot}431_9$
		·2883	$1{\cdot}434_2$
		·2825	$1{\cdot}436_3$
		·2766	$1{\cdot}438_7$

Temperature..... 13·0° C.

regions of the spectrum are given in Tables I. (*a*), I. (*b*), and I. (*c*) respectively, and plotted in fig. 1.

Magneto-Optical Dispersion.

Normal Butyl Alcohol.

The experimental results for the magneto-optical dispersion of butyl alcohol are given in Tables II. (*a*) and fig. 2.

Table I. (*c*).

Propionic Acid.

Spectrometer Determinations.		Photographic Determinations.	
Wave-length (microns).	Refractive index.	Wave-length (microns).	Refractive index.
6678	$1{\cdot}387_3$	·5105	$1{\cdot}392_5$
5876	$1{\cdot}389_4$	·4507	$1{\cdot}396_2$
·5016	$1{\cdot}393_0$	·4275	$1{\cdot}398_0$
·4922	$1{\cdot}393_6$	·4062	$1{\cdot}400_1$
·4713	$1{\cdot}394_9$	·3861	$1{\cdot}402_6$
·4472	$1{\cdot}396_6$	·3600	$1{\cdot}406_3$
		·3238	$1{\cdot}413_2$
		·3118	$1{\cdot}416_3$
		·3039	$1{\cdot}418_5$
		·3008	$1{\cdot}419_6$

Temperature..... 13·4° C.

Fig. 1.

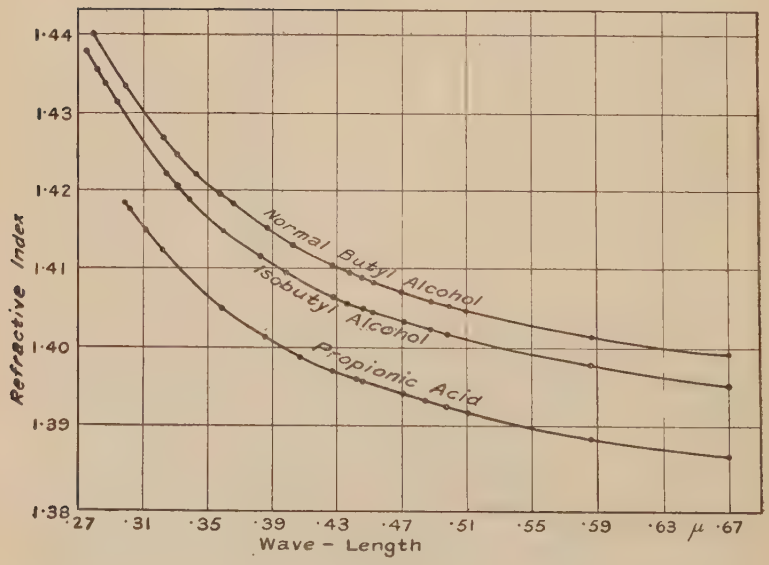


TABLE II. (*a*).

Temp. in degrees Cent.	Wave-length in microns (10^{-4} cm.).	Verdet's constant in mins. per cm. gauss.
12·9	·4545	$·0216_6$
13·0	·4437	$·0228_6$
13·0	·4335	$·0240_7$
12·8	·4250	$·0252_7$
12·8	·4162	$·0264_7$
13·0	·4079	$·0276_8$
12·8	·4007	$·0288_8$
13·0	·3933	$·0300_8$
12·8	·3870	$·0312_9$
13·0	·3808	$·0324_9$
13·0	·3750	$·0337_0$
13·0	·3700	$·0349_0$
13·0	·3646	$·0361_1$
13·0	·3597	$·0372_9$
13·0	·3550	$·0385_1$
12·8	·3503	$·0397_2$
13·2	·3457	$·0409_2$
13·1	·3417	$·0421_3$
13·2	·3380	$·0433_3$
12·8	·3344	$·0445_4$
13·0	·3308	$·0457_4$
13·0	·3272	$·0469_5$
13·0	·3240	$·0481_5$

Table II. (*b*) gives a series of values of n, the refractive index, and δ Verdet's constant for butyl alcohol taken from fig. 1 and Table II. (*a*) respectively. These were made use of in testing the results in relation to Larmor's theory, and it was found that the experimental results could be represented by a formula of the type

$$\phi = n\delta\lambda^2 = \mathrm{K}\left\{\frac{\lambda^2}{\lambda^2 - \lambda_1^2}\right\}^2.$$

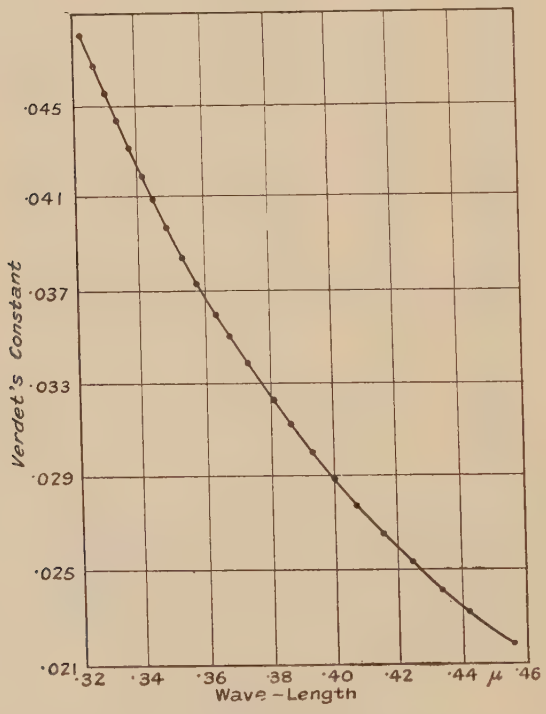
TABLE II. (*b*).

	λ.	n.	δ.
(*a*).....	·4335	$1·411_2$	$·0240_7$
(*b*).....	·4079	$1·413_7$	$·0276_8$
(*c*).....	·3750	$1·417_8$	$·0337_0$
(*d*).....	·3503	$1·422_1$	$·0397_2$
(*e*).....	·3240	$1·427_6$	$·0481_5$

The values of the function ϕ were calculated for each wave-length, and from several pairs of these a set of values of λ_1 and K was obtained.

From (*a*) and (*b*) $\lambda_1 = \cdot 1152\,\mu$ and $K = 5\cdot513 \times 10^{-3}$.
„ (*b*) „ (*d*) $\lambda_1 = \cdot 1151\,\mu$ „ $K = 5\cdot514 \times 10^{-3}$.
„ (*c*) „ (*e*) $\lambda_1 = \cdot 1147\,\mu$ „ $K = 5\cdot520 \times 10^{-3}$.
„ (*a*) „ (*e*) $\lambda_1 = \cdot 1148\,\mu$ „ $K = 5\cdot519 \times 10^{-3}$

Fig. 2.



Normal butyl alcohol.

The mean values of λ_1 and K are $\cdot 1149\,\mu$ and $5\cdot516 \times 10^{-3}$ respectively, δ being measured in minutes per cm. gauss, and λ in microns.

Thus the magneto-optical dispersion of normal butyl alcohol for the region of the spectrum investigated ($\cdot 4545\,\mu$ to $\cdot 3240\,\mu$) can be represented by

$$n\delta = 5\cdot516 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1149)^2\}^2}. \quad . \quad . \quad (A)$$

With values of n read off from fig. 1 δ was calculated for a few wave-lengths where experimental determinations had been carried out. The observed and calculated values of δ are shown in Table II. (*c*).

TABLE II. (*c*).

λ.	δ (calculated).	δ (observed).
·4545 μ	$\cdot 0216_2$	$\cdot 0216_6$
·4250 μ	$\cdot 0251_8$	$\cdot 0252_7$
·3933 μ	$\cdot 0301_2$	$\cdot 0300_8$
·3700 μ	·0348	·0349
·3550 μ	$\cdot 0384_3$	$\cdot 0385_1$
·3417 μ	$\cdot 0421_8$	$\cdot 0421_3$
·3344 μ	$\cdot 0444_9$	$\cdot 0445_4$

Lowry and Dickson *, who determined the magnetic rotations of normal butyl alcohol for a number of wave-lengths in the visible spectrum, compare the rotary powers at various wave-lengths with that at $\lambda = \cdot 5461\,\mu$.

Using values of n obtained from fig. 1 δ was calculated for each of these wave-lengths from equation (A), and a comparison of the results thus obtained with those of Lowry and Dickson is given as follows :—

Wave-length (microns).	Magnetic rotary power relative to that at ·5461 μ.	
	Lowry & Dickson.	Present results.
·6708	·649	·644
·6438	·707	·703
·5893	·849	·849
·5086	1·168	1·167
·4800	1·328	1·327
·4359	1·637	1·648

The value of δ, calculated from equation (A), for sodium light ($n = 1 \cdot 402_4$) is ·01224 at 13·0° C., and assuming ·0131 as the value of Verdet's constant for water at ·5893 μ, the specific rotation of normal butyl alcohol relative to water is therefore ·935.

Isobutyl Alcohol.

The magneto-optical dispersion of isobutyl alcohol has been investigated from ·4610 μ to ·3465 μ. Table III. (*a*)

* Journ. Chem. Soc. i. p. 1072 (1913).

gives the results for the sample which had been subjected to a process of fractional distillation, and Table III. (*b*) the

TABLE III. (*a*).

Temperature (degrees Cent.).	λ (microns).	δ (minutes per cm. gauss).
13·0	·4610	$·0216_6$
13·0	·4495	$·0228_7$
13·1	·4397	$·0240_8$
13·2	·4220	$·0264_8$
13·0	·4133	$·0276_9$
13·2	·3804	$·0337_2$
13·2	·3690	$·0361_2$
13·2	·3640	$·0373_3$
13·1	·3594	$·0385_4$
13·2	·3512	$·0409_5$

TABLE III. (*b*).

Temperature (degrees Cent.).	λ (microns).	δ (minutes per cm. gauss).
13·2	·4610	$·0216_6$
13·2	·4500	$·0228_7$
12·8	·4397	$·0240_8$
13·2	·4307	$·0252_8$
13·2	·4220	$·0264_8$
13·2	·4139	$·0276_9$
12·8	·4063	$·0288_9$
13·0	·3988	$·0301_0$
13·2	·3926	$·0313_0$
13·0	·3864	$·0325_1$
12·8	·3804	$·0337_2$
13·0	·3743	$·0349_2$
12·8	·3693	$·0361_2$
13·2	·3645	$·0373_3$
12·8	·3594	$·0385_4$
12·8	·3551	$·0397_4$
13·2	·3503	$·0409_5$
13·2	·3465	$·0421_5$

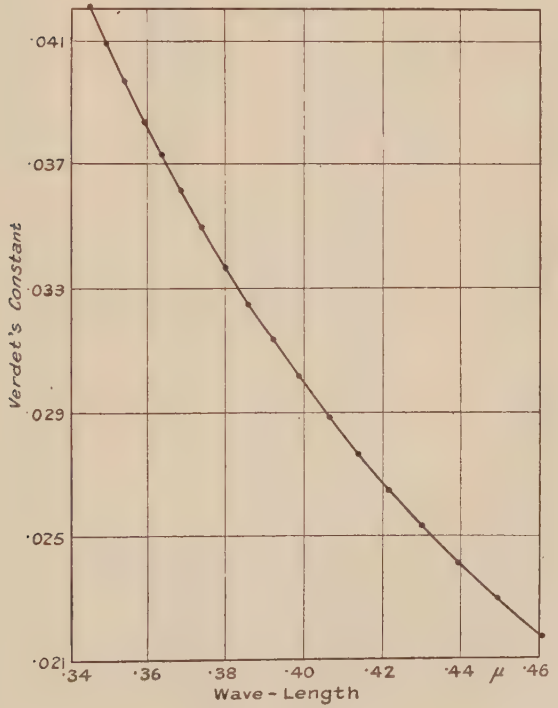
results for the same sample after it had been subjected to further fractional distillation. The results are plotted in fig. 3.

In Table III. (*c*) values of Verdet's constants and the refractive indices corresponding to certain wave-lengths are given, and these values were utilized to calculate the constants of the equation for isobutyl alcohol.

TABLE III. (*c*).

	λ (microns).	*n* (refractive index).	δ (minutes per cm. gauss).
(*a*).....	·4610	$1{\cdot}404_9$	$\cdot0216_6$
(*b*).....	·4397	$1{\cdot}406_4$	$\cdot0240_8$
(*c*).....	·3864	$1{\cdot}412_3$	$\cdot0325_1$
(*d*).....	·3594	$1{\cdot}416_3$	$\cdot0385_4$
(*e*).....	·3551	$1{\cdot}417_1$	$\cdot0397_4$

Fig. 3.



Isobutyl alcohol.

From (*a*) and (*c*) $\lambda_1 = \cdot1162\,\mu$ and $K = 5{\cdot}671 \times 10^{-3}$.
„ (*b*) „ (*d*) $\lambda_1 = \cdot1148\,\mu$ „ $K = 5{\cdot}686 \times 10^{-3}$.
„ (*c*) „ (*e*) $\lambda_1 = \cdot1138\,\mu$ „ $K = 5{\cdot}718 \times 10^{-3}$.
„ (*a*) „ (*e*) $\lambda_1 = \cdot1152\,\mu$ „ $K = 5{\cdot}684 \times 10^{-3}$.

The mean values of λ_1 and K are $\cdot1150\,\mu$ and $5\cdot69\times10^{-3}$ respectively, and the relation between Verdet's constant and wave-length for isobutyl alcohol is represented by

$$n\,\delta = 5\cdot69\times10^{-3}\,\frac{\lambda^2}{\{\lambda^2-(\cdot1150)^2\}^2}. \quad . \quad . \quad (B)$$

This equation was used to evaluate δ for a few wave-lengths at which determinations have been carried out. The result of the comparison between the calculated and experimental values is shown in Table III. (*d*).

TABLE III. (*d*).

Temperature (degrees Cent.).	n.	λ (microns).	δ (observed).	δ (calculated).
13·1	$1\cdot405_7$	·4498	$\cdot0228_7$	$\cdot0229_0$
13·2	$1\cdot408_0$	·4220	$\cdot0264_8$	$\cdot0264_8$
13·1	$1\cdot408_9$	·4136	$\cdot0276_9$	$\cdot0277_3$
13·2	$1\cdot411_5$	·3926	$\cdot0313_0$	$\cdot0312_9$
13·2	$1\cdot415_5$	·3643	$\cdot0373_3$	$\cdot0373_6$
13·2	$1\cdot417_9$	·3507	$\cdot0409_5$	$\cdot0409_6$

TABLE III. (*e*).

λ (microns).	Magnetic rotary power relative to that at ·5461 μ.	
	Lowry & Dickson.	Present results.
·6708	·648	·644
·6438	·699	·703
·5893	·846	·848
·5086	1·162	1·167
·4800	1·321	1·327
·4359	1·635	1·647

The equation was also employed to calculate Verdet's constant for the wave-length of sodium light, and the value obtained is ·01266 at 13·2° C. According to Perkin the specific rotation of isobutyl alcohol for sodium light at a mean temperature of 17·7° C. is 0·9663. Thus, assuming ·0131 as the value of Verdet's constant for water, the value calculated for isobutyl alcohol at 17·7° C. is ·01265. Shwers * obtained ·01266 as the value of Verdet's constant for sodium light at 16° C., and ·01203 at 55·3° C.

* 'Bulletin de la classe des sciences, Académie Royale de Belgique,' p. 719 (1912).

A comparison of Lowry and Dickson's results with the values of the magnetic rotary power relative to that at ·5461 μ, calculated from equation (B), is given in Table III. (*e*).

Propionic Acid.

The magnetic rotations of propionic acid have been examined in the present work from ·4560 μ to ·3170 μ, and

TABLE IV. (*a*).

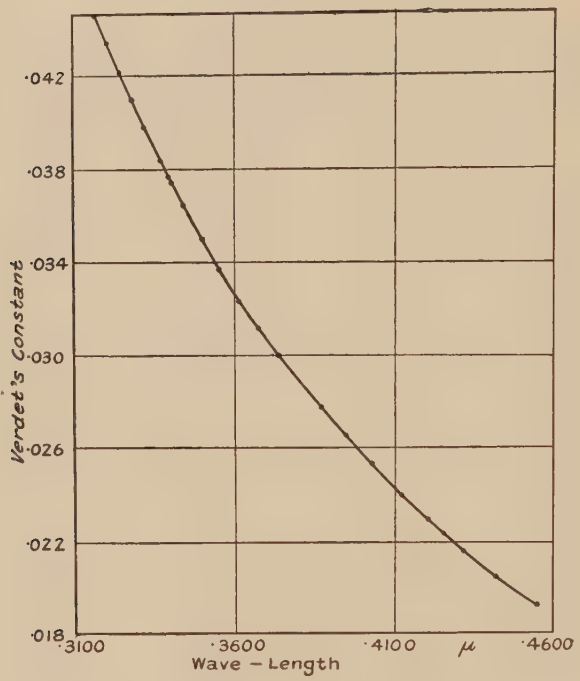
Temperature (degrees Cent.).	λ (microns).	n (refractive index).	δ (minutes per cm. gauss).
12·9	·4560	$1·395_7$	$·0192_2$
12·8	·4437	$1·396_5$	$·0204_2$
13·1	·4326	$1·397_3$	$·0216_2$
12·8	·4274	$1·397_7$	$·0222_2$
12·9	·4225	$1·398_2$	$·0228_1$
13·0	·4128	$1·399_2$	$·0240_2$
13·1	·4037	$1·400_1$	$·0252_2$
12·9	·3958	$1·401_1$	$·0264_2$
13·2	·3883	$1·402_0$	$·0276_2$
13·0	·3811	$1·402_9$	$·0288_3$
12·8	·3739	$1·404_0$	$·0300_3$
13·0	·3680	$1·405_0$	$·0312_3$
13·0	·3618	$1·405_9$	$·0324_3$
12·8	·3568	$1·406_8$	$·0336_4$
12·8	·3513	$1·407_8$	$·0348_3$
13·0	·3458	$1·409_0$	$·0360_3$
12·8	·3411	$1·409_9$	$·0372_4$
13·0	·3391	$1·410_3$	$·0378_4$
12·8	·3370	$1·410_5$	$·0384_4$
13·0	·3324	$1·411_8$	$·0396_4$
12·8	·3285	$1·412_7$	$·0408_3$
12·8	·3247	$1·413_5$	$·0420_5$
13·1	·3205	$1·414_6$	$·0432_4$
12·8	·3170	$1·415_4$	$·0444_5$

the experimental results are given in Table IV. (*a*) and fig. 4. As in the case of the other two organic liquids examined, it was found possible to explain these results by the presence of one strong ultra-violet absorption band.

Table IV. (*b*) contains results read off from fig. 1 and

Table IV. (*a*), and these were made use of in solving the dispersion equation.

Fig. 4.



Propionic acid.

TABLE IV. (*b*).

	λ.	*n*.	δ.
(*a*)	·4326 μ	$1{\cdot}397_3$	$\cdot0216_2$
(*b*)	·4037 μ	$1{\cdot}400_1$	$\cdot0252_2$
(*c*)	·3811 μ	$1{\cdot}402_3$	$\cdot0288_3$
(*d*)	·3458 μ	$1{\cdot}409_0$	$\cdot0360_3$
(*e*)	·3205 μ	$1{\cdot}414_6$	$\cdot0432_4$

From (*a*) and (*c*) $\lambda_1 = \cdot1074$ and $K = 4{\cdot}978 \times 10^{-3}$.
„ (*b*) „ (*d*) $\lambda_1 = \cdot1051$ „ $K = 5{\cdot}002 \times 10^{-3}$.
„ (*c*) „ (*e*) $\lambda_1 = \cdot1037$ „ $K = 5{\cdot}037 \times 10^{-3}$.
„ (*a*) „ (*d*) $\lambda_1 = \cdot1044$ „ $K = 5{\cdot}014 \times 10^{-3}$.
„ (*a*) „ (*e*) $\lambda_1 = \cdot1050$ „ $K = 5{\cdot}007 \times 10^{-3}$.

The mean values of λ_1 and K are ·1051 and $5{\cdot}00_8 \times 10^{-3}$ respectively, and the equation relating Verdet's constant with wave-length is

$$n\delta = 5{\cdot}00_8 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1051)^2\}^2}. \quad \cdot \cdot \quad (C)$$

The accuracy of the representation of the results by the formula (C) is shown in Table IV. (*c*), where the values of Verdet's constant calculated from the formula are compared with the values obtained experimentally.

TABLE IV. (*c*)

λ (microns).	δ (observed).	δ (calculated).
·4560	$\cdot 0192_2$	$\cdot 0192_5$
·4225	$\cdot 0228_1$	$\cdot 0228_3$
·4128	$\cdot 0240_2$	$\cdot 0240_2$
·3958	$\cdot 0264_2$	$\cdot 0264_1$
·3883	$\cdot 0276_2$	$\cdot 0275_9$
·3739	$\cdot 0300_3$	$\cdot 0300_3$
·3680	$\cdot 0312_3$	$\cdot 0312_1$
·3568	$\cdot 0336_4$	$\cdot 0335_9$
·3513	$\cdot 0348_3$	$\cdot 0347_7$
·3411	$\cdot 0372_4$	$\cdot 0372_7$
·3370	$\cdot 0384_4$	$\cdot 0383_5$
·3285	$\cdot 0408_3$	$\cdot 0408_6$
·3170	$\cdot 0444_5$	$\cdot 0444_3$

The value of Verdet's constant for the wave-length of sodium light, calculated from formula (C), is $\cdot 0110_7$. This is in good agreement with that deduced from Perkin's results for propionic acid at various temperatures, after making the temperature correction and assuming ·0131 as the value of Verdet's constant for water. The value thus obtained for Verdet's constant of propionic acid at 13·0° C. for sodium light is $\cdot 0110_6$.

A comparison of Lowry and Dickson's results for propionic acid, with the values of the magnetic rotary power relative to that at $\lambda = \cdot 5461\ \mu$, calculated from equation (C), is given in Table IV. (*d*).

TABLE IV. (*d*).

λ (microns).	Magnetic rotary power relative to that at ·5461 μ.	
	Lowry & Dickson.	Present results.
·6708	·648	·648
·6438	·711	·706
·5893	·856	·850
·5086	1·165	1·165
·4800	1·320	1·321
·4359	1·635	1·633

The Natural Dispersion Equations.

The experimental results have shown that the magneto-optical dispersion of each liquid over the range of spectrum investigated can be explained by the presence of only one absorption band. If the assumption is now made that the natural dispersion over the same range of wave-length can be explained by the presence of the same band, the equation for the natural dispersion now becomes

$$n^2-1=b_0+\frac{b_1}{\lambda^2-\lambda_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

As will be seen later, the refractive indices of normal butyl alcohol, isobutyl alcohol, and propionic acid can be represented within experimental error by equations of the above type, where λ_1 is the wave-length of the absorption band for each liquid as determined from the magneto-optical experiments. The values of b_0 and b_1 can be determined from two values of the refractive index, n_x and n_y, corresponding to wave-lengths λ_x and λ_y.

Normal Butyl Alcohol.

The pairs of values of the refractive indices utilized to calculate b_0 and b_1 are given in Table V. (*a*).

TABLE V. (*a*).

	λ (cm.×10⁵).	*n*.		λ (cm.×10⁵).	*n*.
(A)	6·678	$1·400_2$	(B)	5·876	$1·402_4$
	3·688	$1·418_6$		3·454	$1·423_0$
(C)	5·016	$1·406_2$	(D)	4·922	$1·406_3$
	3·247	$1·427_5$		2·997	$1·434_2$
(E)	4·472	$1·409_9$			
	2·805	$1·441_2$			

From (A) $b_1 = \cdot 888_1 \times 10^{-10}$ and $b_0 = \cdot 9400_3$.
„ (B) $b_1 = \cdot 907_4 \times 10^{-10}$ „ $b_0 = \cdot 9393_8$.
„ (C) $b_1 = \cdot 908_0 \times 10^{-10}$ „ $b_0 = \cdot 9393_0$.
„ (D) $b_1 = \cdot 895_7 \times 10^{-10}$ „ $b_0 = \cdot 9400_0$.
„ (E) $b_1 = \cdot 900_3 \times 10^{-10}$ „ $b_0 = \cdot 9396_0$.

The mean values of b_1 and b_0 are $\cdot 899_5 \times 10^{-10}$ and $\cdot 9396_6$ respectively, and the natural dispersion equation for normal butyl alcohol is

$$n^2 = 1 \cdot 9396_6 + \frac{\cdot 899_5 \times 10^{-10}}{\lambda^2 - (1 \cdot 149 \times 10^{-5})^2}.$$

This equation was used to calculate n for a number of wave-lengths in the region of the spectrum investigated, and the calculated and observed results are given below :—

λ.	n (observed).	n (calculated).
·6678 μ	$1 \cdot 400_2$	$1 \cdot 400_2$
* ·6563 μ	$1 \cdot 4002_7$	$1 \cdot 4004_3$
* ·5893 μ	$1 \cdot 4023_2$	$1 \cdot 4023_6$
·5106 μ	$1 \cdot 405_8$	$1 \cdot 405_7$
* ·4861 μ	$1 \cdot 4071_4$	$1 \cdot 4071_2$
·4713 μ	$1 \cdot 408_2$	$1 \cdot 408_1$
* ·4340 μ	$1 \cdot 4110_8$	$1 \cdot 4110_4$
·3688 μ	$1 \cdot 418_6$	$1 \cdot 418_7$
·2805 μ	$1 \cdot 441_2$	$1 \cdot 441_2$

* The observed values of n, which have been corrected to 13° C., are taken from Landolt and Börnstein, ii. p. 975.

Isobutyl Alcohol.

The constants of the equation for isobutyl alcohol have been calculated from the following pairs of values of the refractive index, taken from fig. 1 :—

TABLE V. (*b*)

	λ (cm. $\times 10^5$).	n.		λ (cm. $\times 10^5$).	n.
(A)	6·678	$1 \cdot 396_3$	(B)	5·876	$1 \cdot 398_8$
	3·337	$1 \cdot 421_1$		2·961	$1 \cdot 431_9$
(C)	4·922	$1 \cdot 403_0$	(D)	4·472	$1 \cdot 406_1$
	2·825	$1 \cdot 436_3$		2·766	$1 \cdot 438_7$

From (A) $b_1 = \cdot 886_3 \times 10^{-10}$ and $b_0 = \cdot 9291_7$.
„ (B) $b_1 = \cdot 904_5 \times 10^{-10}$ „ $b_0 = \cdot 9288_4$.
„ (C) $b_1 = \cdot 887_5 \times 10^{-10}$ „ $b_0 = \cdot 9296_6$.
„ (D) $b_1 = \cdot 887_6 \times 10^{-10}$ „ $b_0 = \cdot 9295_9$.

The mean value of $b_1 = \cdot 892 \times 10^{-10}$ and of $b_0 = \cdot 9293_2$, and the equation for the natural dispersion for isobutyl alcohol is

$$n^2 = 1 \cdot 9293_2 + \frac{\cdot 892 \times 10^{-10}}{\lambda^2 - (1 \cdot 150 \times 10^{-5})^2}.$$

The value of n has been calculated from this equation for various wave-lengths, and a comparison of the observed and calculated results is given below :—

λ.	n (observed).	n (calculated).
$\cdot 6678\,\mu$	$1 \cdot 396_3$	$1 \cdot 396_4$
* $\cdot 6563\,\mu$	$1 \cdot 3963_9$	$1 \cdot 3966_7$
* $\cdot 5893\,\mu$	$1 \cdot 3983_8$	$1 \cdot 3985_9$
$\cdot 5016\,\mu$	$1 \cdot 402_4$	$1 \cdot 402_4$
* $\cdot 4861\,\mu$	$1 \cdot 403_2$	$1 \cdot 4033_2$
$\cdot 4472\,\mu$	$1 \cdot 406_1$	$1 \cdot 406_1$
* $\cdot 4340\,\mu$	$1 \cdot 4071_2$	$1 \cdot 4072_1$
$\cdot 4022\,\mu$	$1 \cdot 410_4$	$1 \cdot 410_4$
$\cdot 3415\,\mu$	$1 \cdot 419_5$	$1 \cdot 419_7$
$\cdot 2883\,\mu$	$1 \cdot 434_2$	$1 \cdot 434_2$
$\cdot 2766\,\mu$	$1 \cdot 438_7$	$1 \cdot 438_8$

* Landolt and Börnstein, ii. p. 975.

Propionic Acid.

The following pairs of values of the refractive indices, taken from fig. 1, were used in calculating the constants of the equation for propionic acid :—

TABLE V. (*c*).

	λ (cm. $\times 10^5$).	n.		λ (cm. $\times 10^5$).	n.
(A)	6·678	$1 \cdot 387_3$	(B)	5·876	$1 \cdot 389_4$
	4·062	$1 \cdot 400_1$		3·861	$1 \cdot 402_6$
(C)	5·016	$1 \cdot 393_0$	(D)	4·922	$1 \cdot 393_6$
	3·600	$1 \cdot 406_3$		3 039	$1 \cdot 418_5$
(F)	4·507	$1 \cdot 396_2$			
	3·008	$1 \cdot 419_6$			

From (A) $b_1 = \cdot 850_3 \times 10^{-10}$ and $b_0 = \cdot 9050_5$.
„ (B) $b_1 = \cdot 866_5 \times 10^{-10}$ „ $b_0 = \cdot 9045_0$.
„ (C) $b_1 = \cdot 870_5 \times 10^{-10}$ „ $b_0 = \cdot 9042_6$.
„ (D) $b_1 = \cdot 878_2 \times 10^{-10}$ „ $b_0 = \cdot 9041_4$.
„ (E) $b_1 = \cdot 892_5 \times 10^{-10}$ „ $b_0 = \cdot 9029_1$.

The mean value of b_1 is $\cdot 871_6 \times 10^{-10}$ and of b_0 $\cdot 9041_7$, and the equation for the natural dispersion is given by

$$n^2 = 1\cdot 9041_7 + \frac{\cdot 8716 \times 10^{-10}}{\lambda^2 - (1\cdot 051 \times 10^{-5})^2}.$$

The value of n has been calculated from this equation for various wave-lengths, and a comparison of the observed and calculated values is given below :—

λ.	n (observed).	n (calculated).
$\cdot 6678\,\mu$	$1\cdot 387_3$	$1\cdot 387_1$
* $\cdot 6563\,\mu$	$1\cdot 3874$	$1\cdot 3874$
* $\cdot 5893\,\mu$	$1\cdot 3894_6$	$1\cdot 3892_6$
$\cdot 5106\,\mu$	$1\cdot 392_5$	$1\cdot 392_5$
* $\cdot 4861\,\mu$	$1\cdot 3942$	$1\cdot 3939$
$\cdot 4472\,\mu$	$1\cdot 396_8$	$1\cdot 396_5$
* $\cdot 4340\,\mu$	$1\cdot 3980$	$1\cdot 3976_1$
$\cdot 4275\,\mu$	$1\cdot 398_0$	$1\cdot 398_2$
$\cdot 3118\,\mu$	$1\cdot 416_3$	$1\cdot 416_1$

* Landolt and Börnstein, p. 974, values corrected to 13·4° C. from values at 0° C. and 20° C.

The Calculation of $\frac{e}{m}$.

The values of $\frac{e}{m}$ for each of the three liquids can be deduced from the experimental results given in the present paper.

According to Larmor's theory :

$$\delta = \frac{e}{2m \cdot C^2} \cdot \lambda \cdot \frac{dn}{d\lambda}. \quad \ldots \quad \text{(E)}$$

It has been shown that the magneto-optical and natural dispersion, over the range of spectrum investigated, can be explained by the presence of the same absorption band, and that the facts can be represented by the following equations:—

$$n \cdot \delta = K \cdot \frac{\lambda^2}{(\lambda^2 - \lambda_1^2)^2} \cdot \quad \ldots \quad \text{(F)}$$

and

$$n^2 - 1 = b_0 + \frac{b_1}{\lambda^2 - \lambda_1^2}. \quad \ldots \quad \text{(G)}$$

Differentiating (G),

$$n \cdot \frac{dn}{d\lambda} = -\frac{b_1 \lambda}{(\lambda^2 - \lambda_1^2)^2},$$

and making use of (E),

$$n\delta = -\frac{e}{2mC^2} \cdot \frac{b_1\lambda^2}{(\lambda^2-\lambda_1^2)^2} \cdot \quad . \quad . \quad . \quad . \quad (H)$$

Therefore, since from equation (F)

$$n\delta = K \cdot \frac{\lambda^2}{(\lambda^2-\lambda_1^2)^2},$$

it follows from equation (H) that

$$\frac{e}{m} = -\frac{2KC^2}{b_1}.$$

K and b_1 have already been determined, so that a value of the ratio $\frac{e}{m}$ may be obtained.

The value of K for butyl alcohol, when δ is expressed in minutes per cm. gauss and the wave-length in microns, is $5{\cdot}516 \times 10^{-3}$. When, however, the value of δ is expressed in radians per cm. gauss, and λ in cm., the value of K is $1{\cdot}604 \times 10^{-14}$.

Then

$$\frac{e}{m} = \frac{2KC^2}{b_1} = \frac{2 \times 1{\cdot}604 \times 10^{-14} \times 9 \times 10^{20}}{{\cdot}8995 \times 10^{-10}}$$

$$= 3{\cdot}21 \times 10^{17} \text{ e.s.u.}$$

$$= 1{\cdot}07 \times 10^{7} \text{ e.m u.}$$

The absolute values of K for isobutyl alcohol and propionic acid are $1{\cdot}655 \times 10^{-14}$ and $1{\cdot}456_7 \times 10^{-14}$ respectively.

The value determined for $\frac{e}{m}$ for isobutyl alcohol $= 1{\cdot}11 \times 10^{7}$ e.m.u., and for propionic acid $\frac{e}{m} = 1{\cdot}00 \times 10^{7}$ e.m.u.

Discussion of Results.

The experimental results for the magneto-optical dispersion of normal butyl alcohol, isobutyl alcohol, and propionic acid can be represented by equations of the type

$$n\delta = K\frac{\lambda^2}{(\lambda^2-\lambda_1^2)^2}$$

where n is the refractive index, δ the value of Verdet's constant, λ_1 the wave-length corresponding to the free period of the resonators, and K a constant whose value

varies from one liquid to another. The values of λ_1 for the three liquids referred to above are ·1149 μ, ·1150 μ, and ·1051 μ respectively.

The values of Verdet's constant for isobutyl alcohol are about 3 per cent. higher than the corresponding values for normal butyl alcohol, but it is interesting to note, in spite of differences in their structural formulæ, that the absorption band responsible for the magneto-optical dispersion of isobutyl alcohol has practically the same wave-length as that for normal butyl alcohol. It is also seen from the experimental results of Jones & Evans and Evans & Evans that the magneto-optical dispersions of propyl and isopropyl alcohols are controlled by absorption bands of practically the same wave-lengths. The results hitherto obtained for the alcohols are collected in Table VI.

TABLE VI.

		λ_1 (microns).	K.	δ for λ = ·5893μ.	Temp. degrees Cent.
CH_4O	Methyl alcohol*.....	·1100	$4·090 \times 10^{-3}$	·00956	9·0
C_2H_6O	Ethyl alcohol†	·1114	$4·96_4 \times 10^{-3}$	$·0113_1$	12·8
C_3H_8O	Normal propyl alcohol*.	·1138	$5·34 \times 10^{-3}$	$·0119_7$	16·0
	Isopropyl alcohol‡.....	·1137	$5·552_6 \times 10^{-3}$	$·0125_2$	13·0
$C_4H_{10}O$	Normal butyl alcohol .	·1149	$5·516 \times 10^{-3}$	$·0122_4$	13·0
	Isobutyl alcohol	·1150	$5·69 \times 10^{-3}$	$·0126_6$	13·2

* Jones & Evans, Phil. Mag. v. (March 1928).
† Stephens & Evans, *loc. cit.*
† Evans & Evans, Phil. Mag. viii. (August 1929).

The Table shows that the values of λ_1 and K increase as molecular weight of the alcohol increases, and the position of the absorption band is displaced on the average about ·0016 μ towards the red by the addition of CH_2 to the molecular weight.

The experimental results also indicate that the refractive indices of normal butyl alcohol, isobutyl alcohol, and propionic acid can be represented within experimental error by equations of the type

$$n^2-1=b_0+\frac{b_1}{\lambda^2-\lambda_1^2},$$

where the constants b_0 and b_1 depend on the nature of the liquid, and λ_1 is the wave-length calculated for a particular liquid from the magnetic rotation experiments.

Evidence as to the nature of the resonators controlling the magneto-optical and ordinary dispersion in the visible

and ultra-violet regions of the spectrum was obtained by combining the results of the magnetic rotation experiments with those relating to ordinary dispersion. The calculations show, in conformity with the results of other investigators, that the electron is the resonator, but the values of $\frac{e}{m}$ [$1{\cdot}07 \times 10^7$ e.m.u. for normal butyl alcohol, $1{\cdot}11 \times 10^7$ e.m.u. for isobutyl alcohol, and $1{\cdot}00 \times 10^7$ e.m.u. for propionic acid] are decidedly lower than the accepted value $1{\cdot}77 \times 10^7$ e.m.u. The calculations of Drude, Victor Henri, and S. S. Richardson indicate that the number of electrons per molecule is practically equal to the sum of the valencies of all the atoms in the molecule, and it seems as if the interactions between these vibrating electrons are equivalent to an increase in the effective mass of the electron.

Summary.

a. The magneto-optical rotations and refractive indices of butyl alcohol, isobutyl alcohol, and propionic acid have been determined for various wave-lengths in the visible and ultra-violet regions of the spectrum.

b (1). The magneto-rotary dispersion of butyl alcohol and of isobutyl alcohol for the ranges of wave-length investigated can be represented by the equations

$$n\delta = 5{\cdot}516 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1149)^2\}^2}$$

and

$$n\delta = 5{\cdot}69 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1150)^2\}^2}$$

respectively, where n is the refractive index and δ Verdet's constant in minutes per cm. gauss for wave-length λ. The absorption bands which control the magnetic rotation for these two liquids have practically the same wave-length, about $\cdot 1150\ \mu$.

b (2). The natural dispersion of butyl and isobutyl alcohols for the ranges of wave-length investigated can be represented by the equations

$$n^2 = 1{\cdot}9396_6 + \frac{\cdot 899_5 \times 10^{-10}}{\lambda^2 - (1{\cdot}149 \times 10^{-5})^2}$$

and

$$n^2 = 1{\cdot}9293_2 + \frac{\cdot 892 \times 10^{-10}}{\lambda^2 - (1{\cdot}150 \times 10^{-5})^2}$$

respectively, where n is the refractive index and λ the wave-length measured in cm. The wave-lengths of the absorption bands are the same as in the corresponding magneto-optical equations.

c. The magneto-rotary and natural dispersions of propionic acid for the range of wave-length investigated can be represented by the equations

$$n\delta = 5{\cdot}00_8 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1051)^2\}^2}$$

and

$$n^2 = 1{\cdot}9041_7 + \frac{{\cdot}871_6 \times 10^{-10}}{\lambda^2 - (1{\cdot}051 \times 10^{-5})^2}$$

respectively, the wave-length of the absorption band being $\cdot 1051\,\mu$ in each case.

d. From the experimental results for the magneto-optical and natural dispersions of normal butyl alcohol, isobutyl alcohol, and of propionic acid the value of $\frac{e}{m}$ has been determined for each liquid:

For butyl alcohol $\frac{e}{m} = 1{\cdot}07 \times 10^7$ e.m.u.; for isobutyl alcohol $\frac{e}{m} = 1{\cdot}11 \times 10^7$ e.m.u; and for propionic acid $\frac{e}{m} = 1{\cdot}00 \times 10^7$ e.m.u.

May 1930.

LXXIII. *The Lateral Bending of Bars, limited by Surfaces of Second Order. By* Prof. A. N. Dinnik, *Member of Academy of Sciences in Ukraina, and* Prof. A. S. Lokshin, *Mining Institute, Dnepropetrowsk* *.

§ 1. *Preface.*

THE lateral bending of bars of variable cross-section presents a very interesting problem. These bars are often used in engineering constructions, on account of their lightness in comparison with uniform bars. The calculation of such bars is not sufficiently understood—as is, for instance, seen from the last edition of 'Hütte' †.

* Communicated by R. V. Southwell, F.R.S.

† 'Hütte,' vol. i. pp. 630-631 (Moscow, 1929).

Mathematically this problem requires the integration of a differential equation of the second order with variable coefficients, by obtaining a transcendental equation to define the critical force in terms of the boundary conditions, and by calculating the roots of the last equation. We have several solutions for special cases of different form of bars. It is desirable here to make a more general investigation ; to take a sufficiently general law of varying cross-section, solve this problem, obtain numerical results, and examine the cases which may be useful for technics. It has been performed in cases where the moment of inertia of the section varies along the axis according to a power or exponential law *.

In this paper we consider the question of the lateral bending of bars limited by second order surfaces, when the line of action of the compressive force coincides with the axis of the surface.

The problem of stability of bars limited by a surface of revolution of the second order has been solved by Lagrange † in a very original way. Lagrange considered only the case in which the ends are supported, and he has not calculated any table of stability coefficients. We examine a general case of second order surfaces, and solve this problem by direct integration of the bending equations. Using the boundary conditions we get in every case the equation for calculating the critical force, and solve it. It is evident that in all cases the critical force may be expressed by a simple formula

$$P = \frac{K.E.I}{l^2},$$

where K is the coefficient of stability, which depends upon the law of variable moment of inertia and upon the means of fastening the ends of the bar,

E, modulus of elasticity,

I, moment of inertia of the largest cross-section of the bar about the bending axis ‡,

l, length of the bar.

* A. Dinnik, " The Lateral Bending of Bars, the Stiffness of which varies by the Binomial Law," ' Iswestia Gornogo Instituta,' 1914 ; " The Lateral Bending of Bars by the Exponential Law," ' Westnik Ingenerow,' 1916 ; " Calculation of Compressed Bars of Variable Section," ' Westnik Ingenerow,' 1929.

† Lagrange, ' Sur la figure des colonnes.'

‡ Excepting the case of a hyperboloid of one sheet, § 5, where I is the moment of inertia of its smallest cross-section.

We calculate the roots of the equations, form tables of stability coefficients "K," and, finally, give applications of the results obtained.

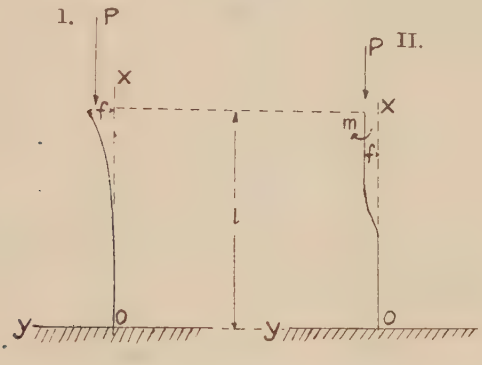
It is evident that the results obtained will be satisfactory provided that the stresses do not surpass the limit of elasticity; this will be true provided that the transverse dimensions of the bar are small in comparison with its length *.

§ 2. *Cylindrical Bar.*

Case I.

The lower end of a vertical uniform bar (for instance, an elliptical cylinder), of length l, is built-in and at the upper free end a vertical force P *is applied* (fig. 1).

Fig. 1.



The bending-moment of the bar at a cross-section distant x from the lower end is

$$M = P(f-y), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where f is the deflexion on the upper end. The differential bending equation is

$$EIy'' + P(y-f) = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and its integral is

$$y = A \cos nx + B \sin nx, \quad . \quad . \quad . \quad . \quad (4)$$

$$n^2 = \frac{P}{ET}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

* In the case of iron or wooden bars, the length "l" must exceed at least 100 times the smallest radius of inertia of the bar's cross-section. *Cf.* Timoshenko, 'Resistance of Materials,' § 175 (1929).

The boundary conditions of the lower and upper ends of the bar,

$$x = 0, \quad y = 0, \quad y' = 0, \ldots \quad (6)$$

$$x = l, \quad y = f, \ldots \quad (7)$$

lead to the following equation for determining the critical force :

$$\cos nl = 0,$$

from which

$$P_k = \frac{\pi^2 EI}{4l^2} = 2{\cdot}4674 \frac{EI}{l^2}. \ldots \quad (8)$$

Case II.

The lower end of the bar is built-in, the upper end is free to move, but cannot turn (fig. 1, II.).

The bending-moment, the differential equation of the bent bar, and its integral are :

$$M = P(y - f) - m, \ldots \quad (9)$$

$$EIy'' + P(y - f) = m, \ldots \quad (10)$$

$$y = A \cos nx + B \sin nx + f + \frac{m}{P}, \ldots \quad (11)$$

where m is the reaction moment which prevents the upper end from turning.

The boundary conditions,

$$x = 0, \quad y = 0, \quad y' = 0, \ldots \quad (12)$$

$$x = l, \quad y = f, \quad y' = 0, \ldots \quad (13)$$

give the following equation for determining the critical force :

$$\sin nl = 0, \ldots \quad (14)$$

from (14) and (4),

$$P_{cr.} = \frac{\pi^2 EI}{l^2} = 4{\cdot}8696 \frac{EI}{l^2}. \ldots \quad (15)$$

§ 3. *Ellipsoidal Bar.*

Let a bar have the form of a triaxial ellipsoid (fig. 2), the equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \ldots \quad (16)$$

The moment of inertia of the largest section in the middle of the bent ellipsoid (in the plane zOy) is

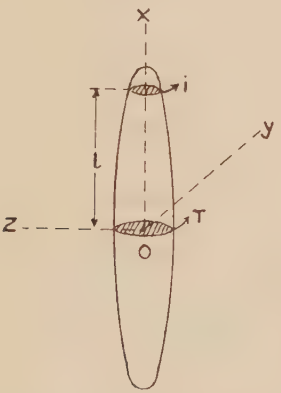
$$I = \frac{\pi b^3 c}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The moment of inertia of a section distant x from the centre of the ellipsoid is

$$I(x) = \frac{\pi \beta^3 \gamma}{4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where β, γ denote the semiaxis of the ellipse which we get by intersection of the ellipsoid by a plane $x = \text{const.}$

Fig. 2.



From (16), (17), (18)

$$I(x) = \frac{\pi b^3 c}{4a^4}(a^2 - x^2)^2 = I.\left(1 - \frac{x^2}{a^2}\right)^2 . \quad . \quad . \quad (19)$$

The differential bending equation, by (2) and (19), is

$$EI\left(1 - \frac{x^2}{a^2}\right)^2 y'' + P(y - f) = 0. \quad . \quad . \quad . \quad (20)$$

Integrating, we get

$$y = \sqrt{a^2 - x^2}\{A \cos u + B \sin u\} + f, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$y' = (a^2 - x^2)^{-1/2}\{A[-x . \cos u + \sqrt{n^2 - a^2} \sin u] - B[x \sin u + \sqrt{n^2 - a^2} \cos u]\}, \quad (22)$$

where $$u = \frac{\sqrt{n^2-a^2}}{2a} \log \frac{a-x}{a+x} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

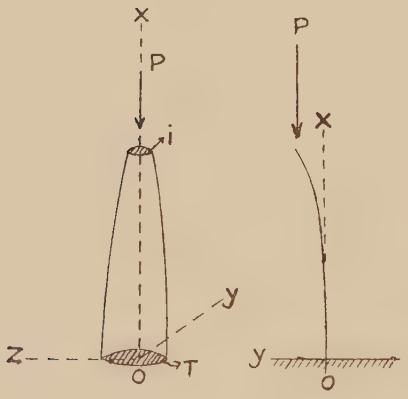
$$. \quad . \quad u^2 = \frac{Pa^4}{EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

as may be verified by inserting (22)–(24) in (20).

We shall consider only the upper half of the truncated ellipsoid. The moment of inertia of the upper section at the distance l from the ellipsoid centre is i ; then the ratio

$$\frac{i}{I} = k^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Fig. .



by $k = 0$ we get an entire ellipsoid,
$k < 1$ „ „ truncated ellipsoid,
$k = 1$ „ „ elliptical cylinder,
$k > 1$ „ „ hyperboloid of one sheet.

Case I.

The lower end is built-in, the upper is free (fig. 3, and fig. 1, I.).

The boundary conditions (5) require that in (21) $B = 0$; condition (6) gives

$$A \cos U = 0,$$

$$U = \frac{\sqrt{n^2-a^2}}{2a} \log \frac{a-l}{a+l} = \frac{\sqrt{n^2-a^2}}{2a} \log \frac{1-\sqrt{1-k}}{1+\sqrt{1-k}}, \quad (26)$$

from which or $A=0$, *i.e.*, rectilinear form of equilibrium, or

$$\cos U = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The smallest root of this equation is $U=\pi/2$. Then, from (24) and (26), we get

$$P_k = (1-k)\left\{1+\left[\frac{\pi}{\log\frac{1-\sqrt{1-k}}{1+\sqrt{1-k}}}\right]^2\right\}\frac{ET}{l^2}=\frac{KEI}{l^2}. \quad (28)$$

The stability coefficient K calculated from (28) is given for different values of i/I in Table I.

TABLE I.

i/I.	K.	i/I.	K.	i/I.	K.
0	1	0·3	2·131	0·7	2·357
0·0001	1·262	0·4	2·205	0·8	2·397
0·01	1·571	0·5	2·262	0·9	2·434
0·1	1·899	0·6	2·313	1	2·467
0·2	2·038				

Taking the second, third, and other roots of the equation (27), we get the second, third, and other values of the critical force.

Case II.

The lower end of the bar is built-in, the upper is free to move but cannot turn (fig. 1, II.).

In this case in (20) we add the term m, and in (21) the term m/P (see § 2 (9)–(11)). The boundary condition (12) gives $A=0$; the condition (13) gives $B=0$ (rectilinear form of equilibrium) or

$$U\tan U=\tfrac{1}{2}\sqrt{1-k}\log\frac{1-\sqrt{1-k}}{1+\sqrt{1-k}}, \quad . \quad . \quad (29)$$

where, calculating U, we find from (24)–(26)

$$P_k = (1-k)\left\{1+\left[\frac{2U}{\log\frac{1-\sqrt{1-k}}{1+\sqrt{1-k}}}\right]^2\right\}\frac{EI}{l^2}. \quad . \quad (30)$$

The critical force is found in the following way :—

(1) We have the ratio of i/I, and from (21) we find k ;

(2) this value of k we put into (19), and calculate the first root U of the equation (26) ;

(3) this value of U we insert in (27), and find the critical force.

In Table II. are given the roots of the equation (29) and the stability coefficients K.

Table II.

i/I.	U.	K.	i/I.	U.	K.
0·0001	2·2088	1·5285	0·5	—	8·1532
0·01	2·5460	2·6637	0·6	3·0617	8·1532
0·1	2·8076	4·5664	0·7	—	8·625
0·2	2·9007	5·6176	0·8	3·1063	9·0643
0·3	—	6·655	0·9	—	9·8700
0·4	3·0004	7·0446	1	3·1416	

§ 4. *Hyperboloid of One Sheet.*

Replacing in (16)–(30) the semiaxis a by ai, we get the solution for a bar having the form of a hyperboloid of one sheet (fig. 4).

Since $i>1$ and $k>1$, we must substitute in all formulas, § 3, $\sqrt{1-k}$ by $i\sqrt{k-1}$ *. Using the known relation between the logarithmical and circular function

$$\log\frac{1-ix}{1+ix}=-2i\tan^{-1}x,$$

we get for the first case, when the lower end is built-in and the upper free, in place of (28),

$$P_k=(k-1)\left\{\left[\frac{\pi}{\tan^{-1}\sqrt{k-1}}\right]^2-1\right\}.$$

The stability coefficient calculated from (21) is given in Table III.

* We denote by $i\sqrt{-1}$, and the moment of inertia of the end section too. We think that this will not cause any confusion.

For the second case (fig. 1, II.), after a not complicated transformation, the equation (29) gives

$$U \tan U = \sqrt{k-1}\tan^{-1}\sqrt{k-1}, \quad . \quad . \quad . \quad . \quad (32)$$

$$P_k = (k-1)\left\{\left[\frac{U}{\tan^{-1}\sqrt{k-1}}\right]^2 - 1\right\}. \quad . \quad (33)$$

Fig. 4.

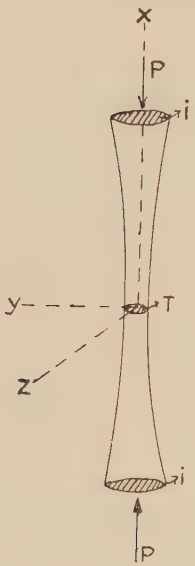


TABLE III.

i/I.	K.	*i*/I.	K.
1	$\pi^2/4$	3	2·879
1·2	2·527	4	3·000
1·4	2·578	5	3·1024
1·6	2·625	10	3·620
1·8	2·674	100	5·233
2	2·724	10000	13·919

The equation (32) has an evident root

$$U_0 = \tan^{-1}\sqrt{k-1},$$

but to this corresponds $P_k=0$. By calculating the next root of the equation (32), and putting it in (33), we obtain Table IV.

Table IV.

i/I.	U.	K.	i/I.	U.	K.	i/I.	U.	K.
1	π	π^2	2	3·254	13·03	10	3·528	26·23
1·2	3·171	10·59	3	3·322	15·40	100	3·906	79·03
1·4	3·196	11·26	4	3·370	17·42	1000	4·212	249·8
1·6	3·217	11·87	5	3·408	·20	10000	4·419	795·0
1·8	3·237	12·45						

If we consider K as depending upon the ratio I/i, *i. e.*, the ratio of the moment of inertia I of the middle smallest section of the one sheet hyperboloid (fig. 4) to the moment i of the largest end section, then we get from the equations (32)–(34) the values in the second column of Table V.

Table V.

I/i.	K.	K.	I/i.	K.	K.
0·0001	0·079	0·030	0·5	6·50	1·79
0·01	0·790	0·284	0·6	7·24	1·95
0·1	2·62	0·856	0·7	7·94	2·09
0·2	3·84	1·17	0·8	8·59	2·22
0·3	4·83	1·41	0·9	9·24	2·35
0·4	5·70	1·61	1		

By using this table we have to substitute in the fundamental formula (1), instead of I, i, *i. e.*, the moment of inertia of the end section. Table V. shows how much the stability of a compressed bar diminishes when it is made thinner in its middle part.

Case III.

The lower end of the bar is supported, the upper end is free to move but cannot turn (fig. 5).

These conditions obtain for each half of a bar which is narrowed at its supported ends and has as boundary a hyperboloid of one sheet (see § 9, fig. 5).

We get the differential equation and its integrals from (19)–(24) by substituting ai instead of a.

$$\left.\begin{aligned} y &= \sqrt{a^2+x^2}\,\{A\cos u + B\sin u\} + f + \frac{m}{P}, \\ u &= \frac{\sqrt{a^2+n^2}}{a}\tan^{-1}\frac{x}{a}. \end{aligned}\right\} \quad . \quad . \quad (34)$$

The boundary conditions are

$$x=0, \quad y=0, \quad y''=0, \quad . \quad . \quad . \quad . \quad (35)$$

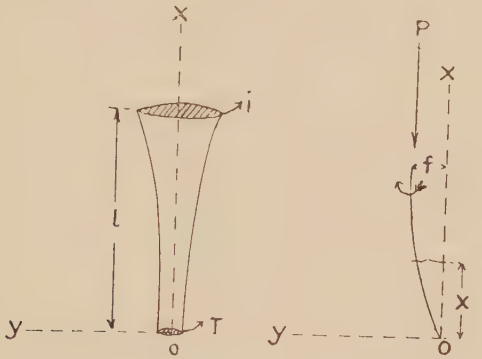
$$x=l, \quad y=f, \quad y'=0; \quad . \quad . \quad . \quad . \quad (36)$$

(35) gives $A=0$; (36) gives

$$U \cot U + \sqrt{k-1} \tan^{-1} \sqrt{k-1} = 0. \quad . \quad . \quad (37)$$

The stability coefficients calculated from (33) and (37) are given in the third column of Table V.

Fig. 5.



§ 5. *Hyperboloid of Two Sheets.*

The bar is limited by a hyperboloid of two sheets (fig. 6), the equation of which is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The moment of inertia of a section at a distance x from the origin of the coordinates is

$$I(x) = \frac{\pi b^3 c}{4a^4}(x^2-a^2)^2. \quad . \quad . \quad . \quad . \quad (39)$$

Substituting (39) in (3), we get

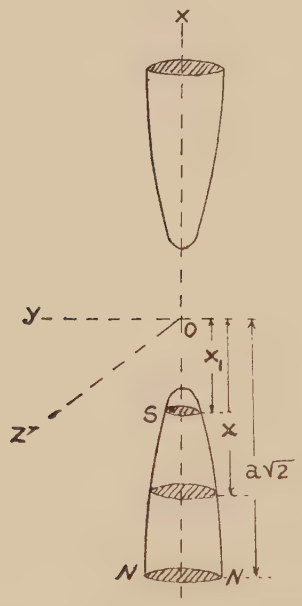
$$(x^2-a^2)^2 y'' + n^2 y = 0, \quad . \quad . \quad . \quad (40)$$

$$n^2 = \frac{4Pa^4}{\pi E b^3 c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The integral of this equation is

$$\left.\begin{aligned} y &= \sqrt{x^2-a^2}\{A\cos u + B\sin u\},\\ y' &= (x^2-a^2)^{-1/2}\{A[x\cos u - \sqrt{n^2-a^2}\sin u]\\ &\qquad + B[x\sin u + \sqrt{n^2-a^2}\cos u]\},\\ u &= \frac{\sqrt{n^2-a^2}}{2a}\log\frac{x-a}{x+a}. \end{aligned}\right\} \quad . \quad . \quad . \quad (42)$$

Fig. 6.



The boundary conditions yield relations by which we can calculate the critical force.

We will consider, for instance, the stability of a part cut off from a one-sheet hyperboloid, where the lower base NN is at a distance $x_0 = a\sqrt{2}$ from the centre of the surface O and the upper SS at a distance X_1. Then

$$I(x_0) = I = \frac{\pi b^3 c}{4}, \quad . \quad . \quad . \quad . \quad . \quad (43)$$

which coincides with the moment of inertia of the largest section of the ellipsoid (17).

Case I. (fig. 1, I.).

Here the boundary conditions (6)–(7) give the following equation for calculating the critical force

$$\left.\begin{aligned} \tan U &= \frac{1}{\sqrt{2}.c}U, \\ c &= \log\frac{\sqrt{1+k}-1}{\sqrt{k}(\sqrt{2}-1)}, \end{aligned}\right\} \quad \ldots\ldots \quad (44)$$

$$P_k = \frac{(c^2+U^2)(\sqrt{2}-\sqrt{1+k})}{c^2}\frac{EI}{l^2}. \quad \ldots \quad (45)$$

The roots of equation (44) and the values of the stability coefficients are given in Table VI.

TABLE VI.

i/I.	U.	K.	i/I.	U.	K.
0·0001	2·2646	0·3941	0·4	1·7110	1·8834
0·01	2·1474	0·7646	0·6	1·6507	2·1122
0·1	1·8995	1·2593	0·8	1·6066	2·3181
0·2	1·8043	1·5540	1	$\pi/2$	$\pi^2/4$

Case II. (fig. 1, II.).

The boundary conditions (12)–(13) give the equation

$$c\left[\sqrt{2(1+k)}+\frac{U^2}{c^2}\right]\tan U = (\sqrt{1+k}-\sqrt{2})U, \quad (46)$$

where

$$c = \tfrac{1}{2}\log\frac{(\sqrt{1+k}-1)(\sqrt{2}+1)}{(\sqrt{1+k}+1)(\sqrt{2}-1)},$$

and P_k is obtained from formula (45).

TABLE VII.

i/I.	U.	K.	i/I.	U.	K.
0·0001	3·3056	0·5756	0·4	3·1440	6·3410
0·01	3·2428	1·4382	0·6	3·1440	7·6044
0·1	3·1780	3·4996	0·8	3·1421	8·8259
0·2	3·1620	4·6550	1	π	π^2

In Table VII. are calculated the roots of (46) and the corresponding stability coefficients.

§ 6. *Elliptical Paraboloid* (fig. 7).

The equation is

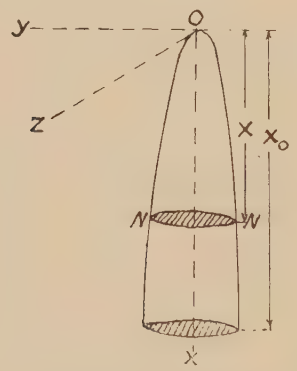
$$\frac{y^2}{2q} + \frac{z^2}{2r} = x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

By (18) and (47) the moment of inertia of the cross-section at a distance x from the paraboloid's top is

$$\mathrm{I}(x) = \pi\sqrt{qr^3}\,x^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

i. e., it changes by the square law.

Fig. 7.



The differential equation of bending and its integral are

$$\left.\begin{aligned} &\mathrm{EI}\left(\frac{x}{l}\right)^2 y'' + \mathrm{P}y = 0, \\ &y = \sqrt{x}\,\{\mathrm{A}\cos\mu\log x + \mathrm{B}\sin\mu\log x\}, \end{aligned}\right\} \quad . \quad . \quad (49)$$

where

$$\mu^2 = \frac{\mathrm{P}l^2}{\mathrm{EI}} - \frac{1}{4}, \qquad \frac{i}{\mathrm{I}} = k^4.$$

The boundary conditions for the first case give *

$$\tan 2\mu\log k = 2\mu, \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and for the second case

$$2\mu\log k = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (50\,a)$$

* The integral has a form (49) when μ is real and different from 0. When μ is imaginary and $\mu=0$, the integral has another form. For detail see the work of A. Dinnik mentioned in § 1.

In both cases

$$P_k = \frac{(1+4\mu^2)(1-k^2)^2}{4}\frac{EI}{l^2} = \frac{KEI}{l^2} \quad . \quad . \quad . \quad (50\,b)$$

In Table VIII. are given the roots of equation (50) and the stability coefficients for cases I. and II.

TABLE VIII.

i/I.	U.	K.	K.
0	0	0·25	0·25
0·1	3·25	1·35	3·60
0·2	4·46	1·59	4·73
0·3	5·79	1·76	5·62
0·4	7·44	1·90	6·39
0·5	9·66	2·02	7·07
0·6	12·90	2·13	7·70
0·7	13·23	2·22	8·29
0·8	28·78	2·31	8·88
0·9	60·27	2·39	9·37
1	∞	$\pi^2/4$	π^2

§ 7. *Elliptical Cone.*

When the top of the cone is at the origin of the co-ordinates the equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

By (18) and (51) the moment of inertia of the section at distance x from the top is

$$I(x) = \frac{\pi b^3 c}{a^4}x = I\frac{x^4}{h^4} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

It varies by the law of fourth power. Here I is the moment of inertia of the section at the distance h from the top.

The differential equation of bending and its integral are

$$\left.\begin{aligned} &EI\left(\frac{x}{h}\right)^4 y'' + Py = 0, \\ &y = x\left\{A\cos\frac{Uh}{x} + B\sin\frac{Uh}{x}\right\} \end{aligned}\right\}, \quad . \quad . \quad (53)$$

where

$$U^2 = \frac{Ph}{EI}, \qquad \frac{i}{I} = k^3.$$

The boundary conditions give for the first case

$$\tan\frac{(1-k^2)U}{k^2} = -U, \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and for the second case

$$(k^2+U^2)\tan\frac{1-k^2}{k^2}U = (l-k^2)U, \quad . \quad . \quad . \quad (55)$$

where in both cases

$$P_k = (l-k^2)^2U^2\frac{EI}{k^2} = \frac{KEI}{l^2}. \quad . \quad . \quad . \quad (56)$$

The roots of equations (54) and (55), with the corresponding K *, are given in Table IX.

TABLE IX.

i/I.	U.	K.	U.	K.
0	0	0	0	0
0·1	2·51	1·20	4·18	3·32
0·2	3·70	1·50	6·44	4·56
0·3	5·03	1·71	9·03	5·50
0·4	6·68	1·87	12·27	6·31
0·5	8·89	2·00	16·65	7·02
0·6	12·13	2·12	23·10	7·67
0·7	17·46	2·22	33·71	8·27
0·8	28·00	2·31	54·77	8·83
0·9	54·47	2·39	117·7	9·36
1	∞	$\pi^2/4$	∞	π^2

§ 8. *Hyperbolical Paraboloid.*

There is no question of the lateral bending of an elastic body, limited by such a surface, because of its form. It is, of course, possible to study the stability of a bar partly limited by a surface of a hyperbolic paraboloid

$$\frac{y^2}{q} - \frac{z^2}{r} = 2x,$$

and partly by another surface; for instance, by two planes $y = \pm h$. But this problem is out of our consideration.

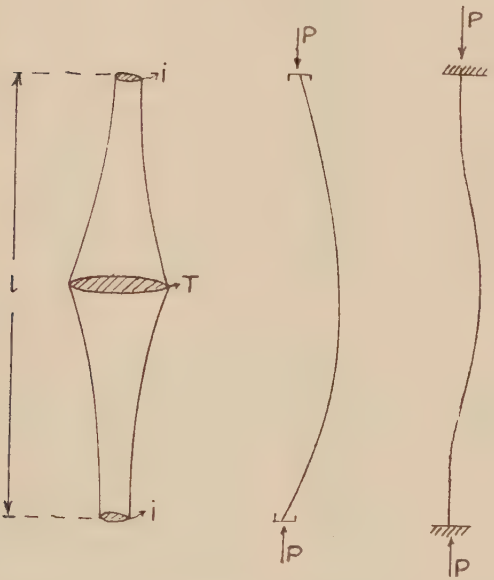
* For other cases of fastening the ends see the work of A. Dinnik mentioned in § I.

§ 9. *Symmetrical Column.*

Let us consider a column symmetrical in relation to its middle and narrowed at both ends. Let each half of this column be a part of a body limited by a surface of second order, as mentioned above. The axis of the surface coincides with the axis of the column.

When both its ends are supported (fig. 8, *b*), half of it is in the conditions of the first case; with the ends built-in

Fig. 8.



(fig. 8, *c*), it is in the conditions of the second case. Hence the critical force for this kind of column is given by formula (1), where I is the moment of inertia of the column's middle section, l = the length of it, K = stability coefficient.

For the ellipsoid and hyperboloid of one sheet K is given in Table X.

When $\frac{i}{I} < 1$ we have an ellipsoid. In particular, when $i = 0$ the ellipsoid is entire (not truncated); when $i = I$ it is an elliptical cylinder; when $i > I$ it is a hyperboloid of one sheet.

For the columns of which each half is a part of a hyperboloid of one or two sheets, or an elliptical paraboloid, or a cone, the coefficients are given for the case of supported ends in Table XI., and for the case of built-in ends in Table XII.

TABLE X.

i/I.	K ends support.	K ends built-in.	Remark.
0	4·00	—	
0·0001	5·10	6·11	
0·01	6·28	10·65	
0·1	7·60	18·27	
0·2	8·15	22·47	
0·3	8·52	—	Triaxial ellipsoid.
0·4	8·81	28·18	
0·5	9·00	—	
0·6	9·25	32·61	
0·7	9·43	—	
0·8	9·59	32·26	
0·9	9·73	—	
1	π^2	$4\pi^2$	Elliptical cylinder.
1·2	—	42·36	
1·4	10·32	45·03	
1·6	10·51	47·49	
1·8	10·69	49·81	
2	10·85	52·13	Triaxial hyperboloid of one sheet.
3	11·50	61·59	
4	12·00	69·66	
5	12·41	76·80	
10	13·87	104·9	
100	20·94	316·1	
1000	33·52	—	

§ 10. *Applications.*

We very often meet in engineering compressed columns of varying section, for instance, in cranes, connecting-rods, and the details of aeroplane construction. Columns of varying cross-section having the stability of uniform columns are much lighter, and are therefore used in practice.

Let us consider some examples.

Example 1.

The lateral bending of an aeroplane pine strut of elliptical form is to be examined. The length $l=200$ cm., the largest middle cross-section is an ellipse with semiaxis $C=6$,

TABLE XI.

i/I.	K ellipt. parabol.	Hyperboloid of two sheets.	Elliptical cone.	Hyperboloid of one sheet.
0	1·00	—	—	—
0·0001	2·00	1·57	0·32	0·12
0·01	3·45	3·06	2·15	1·35
0·1	5·40	5·18	4·81	3·42
0·2	6·37	6·21	6·20	4·67
0·3	7·05	—	6·84	5·64
0·4	7·61	7·53	7·48	6·45
0·5	8·09	—	8·01	7·17
0·6	8·51	8·45	8·47	7·79
0·7	8·89	—	8·87	8·36
0·8	9·26	9·26	9·23	8·90
0·9	9·57	—	9·56	9·40
	π^2	π^2	π^2	π^2

TABLE XII.

i/I.	Elliptical paraboloid.	Hyperboloid of two sheets.	Elliptical cone.	Hyperboloid of one sheet.
0	1·00	—	—	—
0·0001	2·80	2·30	0·66	0·32
0·01	5·84	5·75	4·90	3·16
0·1	14·39	14·00	13·30	10·49
0·2	18·93	18·62	18·23	15·36
0·3	22·49	—	22·02	19·32
0·4	25·54	25·36	25·23	22·80
0·5	28·28	—	28·08	26·00
0·6	30·79	30·44	30·68	28·97
0·7	33·14	—	33·08	31·76
0·8	35·35	35·30	35·33	34·36
0·9	37·47	—	37·46	36·96
1	$4\pi^2$	$4\pi^2$	$4\pi^2$	$4\pi^2$

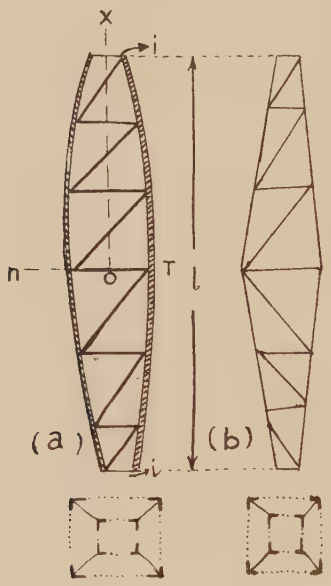
$b=3$ cm. The end sections are ellipses with semiaxes $c_0=4$, $b_0=2$ cm. Young's modulus $E=100000$. Tensile strength, 100 kg./cm.^{2}.

Here the moment of inertia of the middle cross-section about the axis of the smallest stiffness is

$$I = \frac{\pi b^3 c}{4} = 40{\cdot}5\,\pi,$$

and for the end sections $i = 8\pi$, where $i/I = 0{\cdot}198$, or $0{\cdot}2$ approximately. Using Table VIII. for an ellipsoid with supported ends when $i/I = 0{\cdot}2$, we have the stability coefficient $K = 8{\cdot}15$, whence, from (1), $P = 2600$ kg.

Fig. 9.



The compressive stress in the end sections is 103 kg./cm.^{2}, *i. e.*, exceeds a little the permissible stress. The stress in the middle cross-section will evidently be smaller.

If the strut had the form of an elliptical cone, we should get from Table IX. $K = 6{\cdot}20$ and $P = 1980$ kg.

Example 2.

Lattice column of parabolic form.

Let a column be constituted of four angle-irons of constant sections joint with a sufficiently solid lattice (fig. 9). The

moment of inertia of a section at the distance x from the middle of the column is

$$I(x) = 4I_c + 4S\eta^2, \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where I_c is the moment of inertia of the angle-iron relatively to its centre of gravity, S the area of every angle-iron, η the distance of the centre of gravity from the neutral plane of the bent column.

It frequently happens that the first term in (57) is very small in comparison with the second. Neglecting it, we have

$$I(x) = 4S\eta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The angle-irons are curved in the form of a parabola

$$\eta = A\frac{a^2 - x^2}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Hence the largest moment of inertia (at the middle of the column) is

$$I = 4SA^2.$$

The moment of inertia at any other section is

$$I(x) = I \cdot \left(1 - \frac{x^2}{a^2}\right)^2, \quad . \quad . \quad . \quad . \quad (60)$$

i. e., it varies by the same law as the ellipsoid; so we can calculate this column, using Table X.

Example 3.

A crossbar of 7 m. long consists of 2 U-irons, no. 12, curved in the form of a parabola. The distance between the U-irons is in the middle 10 cm. and at the ends 1 cm. A sheet of 1 cm. is pressed between the U-irons.

The moment of inertia of the middle section * is

$$I = 2(92 + 17{\cdot}3.6{\cdot}65^2) = 1710.$$

For the end sections

$$i = 2(92 + 17{\cdot}3.2{\cdot}15^2) = 344.$$

* The dimensions are taken by the U.S.S.R. standards by 'Hütte,' i. p. 855 (1929).

Hence

$$i/\mathrm{I} = 0{\cdot}2.$$

From Table VIII. we have

$$\mathrm{K} = 8{\cdot}15,$$

and the critical force from (1)

$$\mathrm{P}_k = \frac{8{\cdot}15 \,.\, 2{\cdot}10^6 \,.\, 1710}{490000} = 5700.$$

The compressive stress is 1670 kg./cm.^{2}, which is smaller than the elastic limit.

Example 4.

Pyramidal column latticed.

A very important case in practice is when the edges of the column have a rectilinear form (fig. 9, *b*). In this case the moment of inertia varies according to the same law as the elliptical paraboloid, § 7. Hence, in calculating this column we may use the values of the second column of Tables XI. and XII.*

Example 5.

A supporting leg of an overshaft pile-driver of 20 m. long is compressed by a force of 30 t. The piped section is of four equilateral angle-irons tied together by a lattice. The allowed intensity of compressive stress is 500 kg./cm.^{2}. The reserve of stability in lateral bending is to be 4. The dimensions of the angle-irons and the distance between them have to be found.

At first we calculate the simple compression of the leg, and find that the area of every angle-iron is 15 cm.^{2}. From the standard tables it is seen that this corresponds to the angle-irons no. 8, *i. e.*, 80, 80, 10 mm. ; its area, 15·11 cm.^{2}. The intensity of compression is 496 kg./cm.^{2}. Now we have to get the distance between the angle-irons so that the leg should be sufficiently stable. If we take the width of the leg as 30 cm., the moment of inertia of the cross-section of the leg, from (57), is

$$\mathrm{I} = 4.87{\cdot}2 + 4.15{\cdot}11.12{\cdot}66^2 = 10000,$$

* We neglect the value I_c with comparison $\mathrm{S}\eta^2$. The calculations made by student D. B. Volper show that, even in a most unfavourable case, when at the ends of the column the angle-irons touch one another with their shelfs, the error attains only a few per cent.

where the moment of inertia of the angle-iron about the axis passing through its centre of gravity is 87·2. The distance of the centre of gravity of the angle-iron from its edge is 12·66 = 15 − 2·34.

All values are taken from the standard tables.

If the section is constant, the critical force by Euler's formula is P = 49500, *i. e.*, the stability reserve is less than twice.

To increase the stability the leg must be made larger in the middle, the end sections being kept the same. We place the angle-irons on the half-length of the leg at a distance of 50 cm. Then the moment of inertia of this section, constituted from four angle-irons no. 8 at a distance of 50 cm., is

$$I = 4.[87{\cdot}2 + 15{\cdot}11.22{\cdot}66^2] = 31400.$$

The ratio $i/I = 0{\cdot}31$.

In our case the moment of inertia varies along the leg by a law sufficiently near to the square law. The stability coefficient from Table IX., when $i/I = 0{\cdot}3$, K = 7·05, and the critical force by (1),

$$P_k = 110000 \text{ kg.},$$

that is, by widening the leg in the middle while the angle-irons no. 8 remain the same its stability increases more than twice, and we get the reserve stability in more than three times.

But that is not enough; we put the width of the leg in the middle 52 cm.; then, by (51),

$$I = 4.87{\cdot}2 + 4.15{\cdot}11.25{\cdot}68 = 34200.$$

The ratio $i/I = 0{\cdot}29$, or approximately 0·3, which corresponds with Table XI, K = 7·05. Hence, by (1), P = 120000 kg., *i. e.*, the required reserve of stability.

If the strut's flanges are curved in the form of a parabola (59), or of a broken line the tops of which lie on the parabola (59), then the moment of inertia $I(x)$ of the strut varies by the law (60). From Table X., when $i/I = 0{\cdot}3$, the stability coefficient K = 8·52;

$$\text{by } I = 31000 \text{ the critical force} = 133000,$$

$$I = 34200 \quad \text{„} \quad \text{„} \quad \text{„} \quad = 14500,$$

i. e., increases approximately about 20 per cent.

Example 6.

A connecting-rod of a low-speed steam engine has a length of $l=554$ cm. It is narrowed to the ends like a cone; the largest diameter of the connecting-rod is 22 cm., the smallest is 18 cm. The critical force has to be found. Here the moment of inertia $i=5150$, $I=11500$, where $i/I=0{\cdot}45$. From Table IX. $K=7{\cdot}75$ (average between 7·48 and 8·01), and by (1) $P=582000$. The critical compression in the narrowest place is 2280 kg., *i. e.*, it does not exceed the elastic limit for steel *.

§ 11. *Conclusion.*

We have examined in this paper the question of lateral bending of bars of varying cross-section which have the form of one of the surfaces of second order when the axis coincides with the axis of the surface.

We have considered the elliptical cylinder, elliptical cone, triaxial ellipsoid, hyperboloid of one and of two sheets, and elliptical paraboloid. The hyperbolical paraboloid and the sphere are out of the question, because of the form of their surface. We have integrated the differential equations of bending, and from the boundary conditions we have obtained transcendental equations, from which we have determined the critical forces.

It appears that in all these cases the critical force may be expressed by a simple formula,

$$P_{cr.}=\frac{KEI}{l^2}.$$

The stability coefficient K depends only upon the form of the bar and upon the means of fastening it.

We have given tables for the coefficients K.

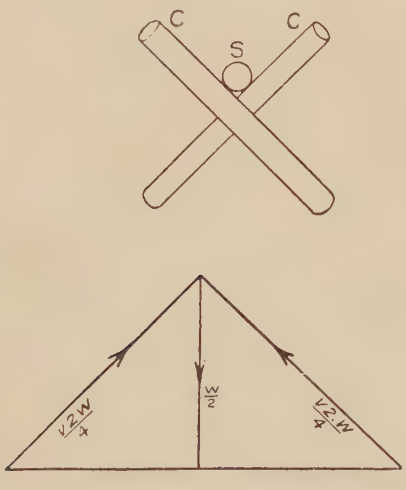
At the end we have given several examples dealing with the practical applications of the results obtained.

* The dimensions are taken for a lift of one of the mines of Donbass. We neglect the influence of the heads of the connecting-rod because they are small. The diameter of the cylinder of the lift is 107 cm.; the steam-pressure is 7 atm., which corresponds with a loading on the connecting-rod about 65 T., *i. e.*, the stability reserve is about 9.

LXXIV. *Friction of Dry Solids* in vacuo. *By* Prof. P. E. Shaw, *M.A., D.Sc., and* E. W. L. Leavey, *M.Sc., University College, Nottingham* *.

THE relation of the friction of dry solids to their elastic constants or other characteristic properties may prove of great theoretical interest. Recently experiments in this connexion have been published by G. A. Tomlinson (Phil. Mag., Suppl., June 1929). We shall endeavour to show in this paper that further relations can be established.

Fig. 1.



To measure the coefficient of friction of dry clean solids we place them in a vacuum tube on which is threaded a heating coil, so that by exhaustion and repeated heating to 350° C. all lubricants as well as all surface strain can be removed from the rubbing surfaces.

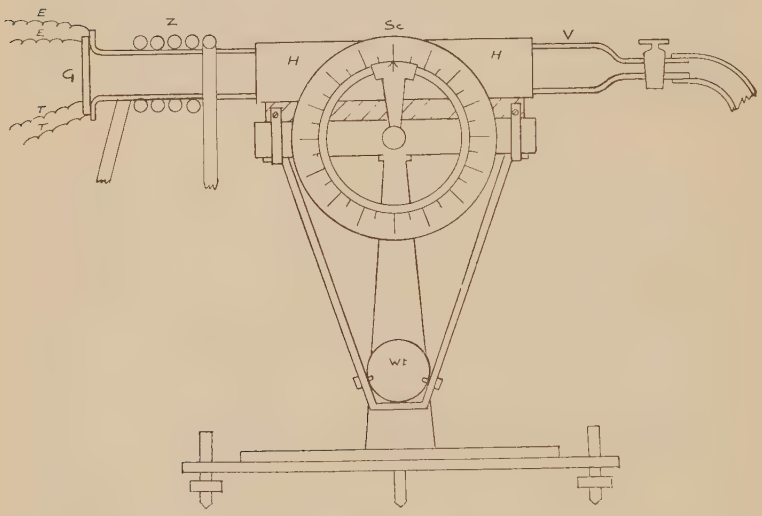
The friction apparatus is a development of that described by one of us (Shaw, Journ. Sci. Inst. iv. April 1927). The general view is shown in fig. 2, and a perspective view showing details inside the vacuum tube is given in fig. 3.

A cylinder of the material is cut into five lengths ; one forms a sliding rod, S, 12 cm. long, the others, each 4 cm.

* Communicated by the Authors.

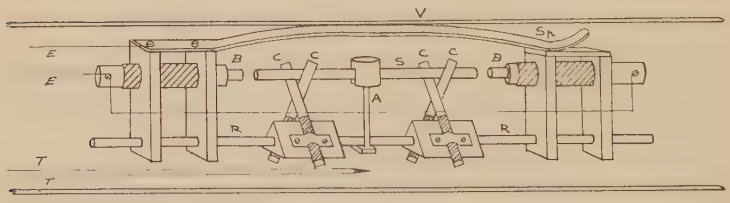
long, are arranged in two crutches, C, C, whose arms are each at 45° to the vertical (fig. 1). The slider rests on the two crutches, and slips when the vacuum tube V (figs. 2 and 3), which turns on a horizontal axis, is tilted to the angle of repose θ of the materials. By reversing the angle

Fig. 2.



The Apparatus.

Fig. 3.



Inside the vacuum tube.

of slope the slider moves the other way, and the true angle of repose is taken as half the difference of the two angular readings. The total weight, W, of the slider is supported equally at four contacts, and as the system is symmetrical the weight supported by one crutch is $\frac{W}{2}$. The reaction

of each arm of the crutch is normal to its length and equal to $\frac{\sqrt{2}W}{4}$ (see fig. 1). As there are four of these reactions the total normal thrust when the slider is tilted through angle θ is $\sqrt{2}W.\cos\theta$. The total component force urging the rod to slip being $W.\sin\theta$, we have, by the law of friction,

$$W\sin\theta = \mu(\sqrt{2}W.\cos\theta);$$

hence

$$\mu = \frac{\tan\theta}{\sqrt{2}},$$

where μ=coefficient of friction.

Since friction may depend on surface strain, and so on previous rubbing, it is clearly an advantage to insure that the same parts of slider and crutches are rubbed throughout an experiment, otherwise readings would not progress regularly. For this reason only one degree of freedom—that along its length—is allowed to the slider. This is arranged by attaching to the slider an arm, A, weighted, to rub *lightly* against the fixed rod, RR, which is parallel to the axis of the tube, and to which all the working parts except the slider are attached. The crutches are shown insulated by mica sheets from the rest of the apparatus. During the experiments the sliding is not watched, but is registered by electric contact. At each end of the sliding rod, and free of it, is a stop B; and when the slider strikes B at either end it completes a circuit and excites a current indicator outside the vacuum tube. There are advantages in this electric touch method, especially as the heating coil H is usually in place and obscures the view of the inside of the tube. Thus the observer does not need to observe the actual movement of the slider. Each of the stops BB is a buffer having a bar sliding in a tube with a steel spring attached behind it. It is important to have BB yielding, not rigid, for two reasons: (*a*) If the slider meets a rigid stop when moving down a slope it jumps and falls again, and so the surfaces are bruised and their friction modified just where the friction is to be tested. But the buffers bring the slider slowly to rest and obviate shock. (*b*) To obtain a true coefficient of friction the slider should touch the crutches, and no other solid such

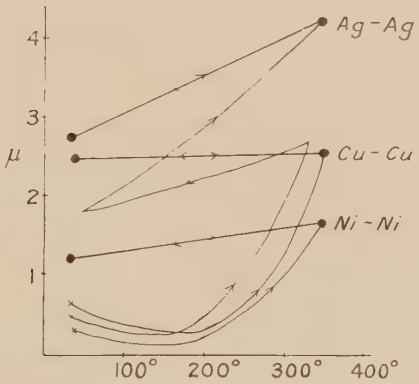
as the stops, at the moment when friction begins ; for any such contact may admit some spurious stress normal to the surfaces. This would introduce error in the coefficient of friction. To explain the action of the buffer, suppose the apparatus is tilted and the slider is compressing a buffer spring. When the tilt of the apparatus is slowly reversed the buffer recovers from the compression and drives the slider out of contact with it. The indicator reveals that contact is broken, and when the apparatus is further tilted to the angle of repose on the reverse side the rod slides again and moves to the lower buffer, where electric contact again occurs. Thus we insure free friction for the slider with one degree of freedom only.

Alternating current is used in the heating coil to avoid magnetization of the iron or nickel rods, and consequent error in μ. As a further precaution demagnetization was secured by reducing the A.C. gradually to zero, in the usual way. The results were then checked by repeating the whole experiment, using Bunsen burners in place of the heating coil. The temperature is recorded by a thermo-couple, insulated in the vacuum tube, and connected to a calibrated galvanometer outside. The leads for the contact arrangement are E E, and those for the thermo-couple T T. These all leave the tube where it is sealed to the glass plate G, sealing-wax being used. As the temperature of the seal must be kept low, a few turns of rubber tube Z carrying cold water are coiled on the vacuum tube. Vacuum to about $\frac{1}{100}$ mm. is used. A tap and rubber tube connexion to a pump are shown on the right. The tube and its connexions are free to swing either way 90° to the horizontal, and the angle is read on the scale Sc. To attain balance on the tilting apparatus a counterpoise Wt is attached below. The frame carrying the components inside the vacuum tube is fitted tight by the steel spring Sp, so that even when the tube is set vertical nothing but the slider can move.

The metals used (Fe, Ni, Cu, Ag, Al) are cleaned by finest sand-paper and then by a clean blade. The soda-glass rods used are given a preliminary cleaning in a Bunsen flame. The friction between each of the various combinations of these materials is found first at ordinary temperature and pressure. Then vacuum is attained, and temperature raised to the upper limit and lowered again repeatedly, the values of the angle of repose being recorded

throughout until readings at any temperature are constant. Before the first heating the coefficient of friction is invariably low, between 0·1 and 0·3, and remains low after prolonged evacuation and sliding to and fro. Fig. 4 shows, for three cases, how μ changes as heating and sliding proceed. When heating commences μ falls to a minimum at about 150° C. (see fig. 4). Then, as temperature rises, it rises slowly till about 250°, then rapidly to the limiting temperature 350°. After repeated heating and cooling μ attains a constant high value. Some of our values point to the failure of the law of friction at high temperatures. Thus, for Ag–Ag and Ni–Al at 320°, μ is about 4, so that the angle

Fig. 4.



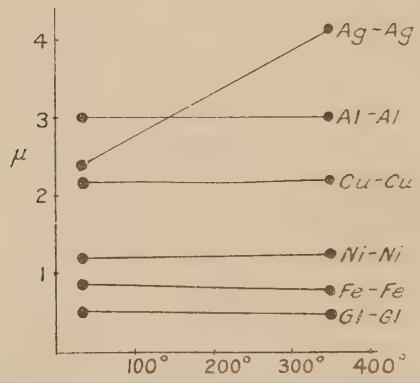
Variation in μ as the heat treatment proceeds for three like pairs. Initial readings shown by star. Final values for the temperature range shown between circles.

of repose is about 80°. In some cases we find the slider even becomes vertical without sliding, so that here the tangential force is finite and the normal force vanishes and $\mu = \infty$. This corresponds to Tomlinson's experience with lead (*loc. cit.* p. 936), and indicates that the two surfaces are strongly combined. Even steel, which normally has a low coefficient of friction, behaves in the same way in some circumstances; for when two Johansson gauges are "wrung" together the force required to pull them apart may amount to several kilograms to the square centimetre. Under such conditions the law of friction could not be applied.

Fig. 5 shows the values of μ for like materials depending on temperature, and fig. 6 shows the results when the rubbing materials are unlike. The curves show a large temperature coefficient with Ag–Ag, Ag–Al, Ag–Ni, Al–Fe, and Al–Ni ; other combinations of the metals used have little or no temperature coefficient when once the full values of μ have been attained by vacuum heat.

Direct comparison can be made between our results at ordinary temperature and those given by G. A. Tomlinson (*loc. cit.*) for Cu, Al, Fe, Ni, and glass. He scours the metals with fine emery, then polishes, and cleans with ammonia, alcohol, and ether. In experimenting on glass he uses

Fig. 5.



Final values for like materials. Silver alone has marked temperature coefficient.

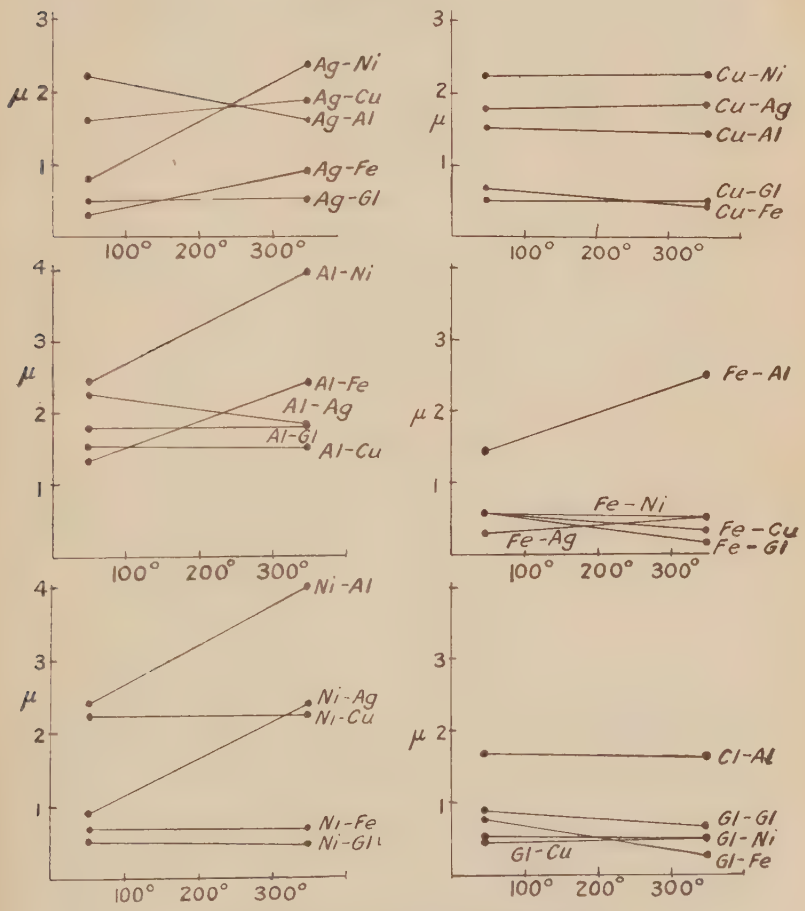
a fused bead on a plate of glass. Friction is then measured in the usual way by applying an increasing horizontal load to the upper solid till it slides on the lower. No heat is applied, and the operation is performed in open air.

Thus the methods employed in this research differ markedly from ours ; the great disparity between his results and ours is shown in fig. 7, though it will be observed that in both sets of results soft metals have large friction *.

* It may be noted, as a practical issue of our results, that in bearings which become hot after lubricant has failed, friction increases rapidly when the temperature rises above the temperature of relaxation : for then the low values of μ usually found rapidly increase about three-fold to our values. This would precipitate "seizing."

For metals our results are higher, for glass lower, than his. In the case of glass this is not surprising; for the value of μ changes from say 0·3 for recently heated glass, to, say, 1·0 or more after prolonged treatment with boiling

Fig. 6.



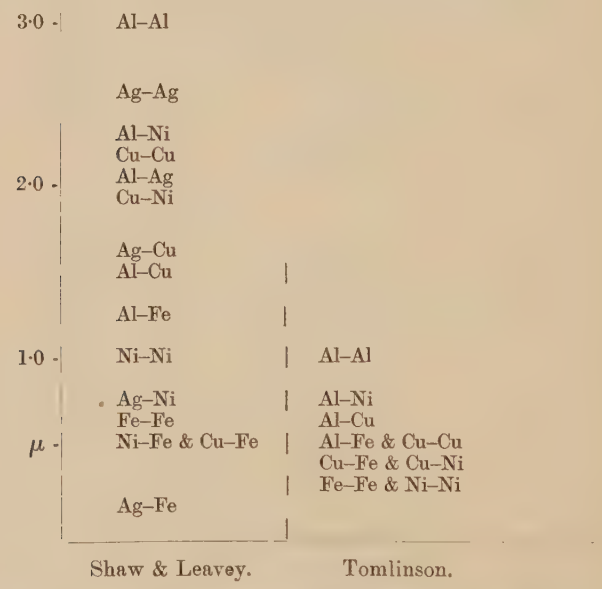
Final values for the five metals and glass.

chromic acid, which roughens and disintegrates the surface. Various glasses vary enormously in composition. Unless therefore glass of the same composition and treatment be used, it is useless to compare the results. But in the case

of the metals, which would all be approximately pure, and are in each case treated not chemically but mechanically by clean emery or knife, the difference between the results of the two researches is striking.

We attribute the disparity primarily to the fact that the vacuum-heat treatment is very drastic. It not only at 350° removes adsorbed layers or adherent impurity, but, what seems much more important, fundamentally modifies

Fig. 7.



First column.—Our values of μ, at ordinary temperature, after heat treatment.
Second column.—Tomlinson's values of μ.

the metal surfaces, relaxing surface strain and leaving the surfaces annealed *.

There are good grounds for considering that at or below 300° C. surface relaxation occurs; Sir. G. Beilby (see 'Aggregation and Flow of Solids') found that at some

* It might be suggested that at high temperature oxides may form on the metals, even in the good vacuum used, and modify the friction. But close observation in every case tried by us revealed no oxide, the rubbing surface after the experiments being quite bright.

temperature below 300° " vitreous " metal surfaces change to the crystalline state, and that any peculiar thermo-electric quality produced by surface strain is destroyed. Again, in a research by one of us (Shaw, Proc. Roy. Soc. A, xciv. 1917) it is shown that the surfaces of many metals (Pb, Al, Zn, Fe, Ni, Au, Pt) undergo triboelectric transformation at some temperature, between 100° and 300°, peculiar to the material.

The difference between cold-worked and annealed metals is seen in a variety of ways ; mercury wets silver and copper better when hardened than when annealed ; also iron is more soluble in acids in the former state than in the latter.

We have applied various tests to our theory—

1. The surfaces have been heated to 300° in contact with one another and values of the coefficient of friction taken at successive temperatures rising and falling. In this way the surfaces would press one another over the whole range of temperature. This is the standard method, whose results are shown above.

2. The surfaces are heated when out of contact and allowed to cool before friction is measured. The coefficients in this method are slightly higher than in the former. Thus for the metals tried the value of μ is about 10 per cent. higher than by method 1. This we expected, as the rubbing of the surfaces when warm would impose some strain on them and harden them.

3. The surfaces having been annealed in vacuum, giving the high values shown above, are then removed from the vacuum, treated with finest emery, then polished and cleaned as in Tomlinson's experiment. The values of μ are now found as low as his.

4. The results so far are for light loads (say 15–25 gm.), *i. e.*, the rod and slider combined (see fig. 3).

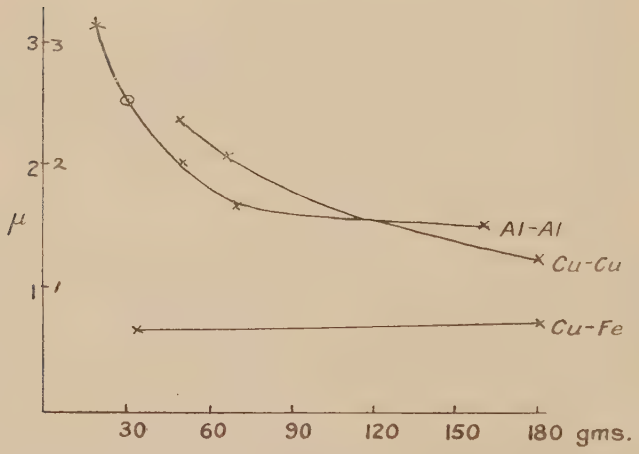
To vary the conditions, larger, and even smaller, loads than these are now tried. In fig. 8 is shown the effect on μ of varying load from 20 gm. to 180 gm. for Al–Al, Cu–Cu, Cu–Fe.

Great reduction of μ occurs for the like metals. For the case of Cu–Fe the value is not materially affected by greater loads. Enough has been done even with these

cases to demonstrate that an annealed surface is liable to change in character when friction occurs under heavy loads. On comparing these results with those of fig. 7 it is seen that, even now, our values are considerably higher than Tomlinson's.

This is not surprising. Tomlinson uses abrasives which cut into the material, dislodging some masses and straining other parts; he next polishes under pressure, and, as we know from Rayleigh's work, polish connotes surface flow. In contrast, friction without previous straining between two metals, as in our work, is a continued process of

Fig. 8.



Values for heavy loads. Commercial metal used except in one case marked ⊙, which is 99·65 per cent. pure.

combination and sundering under pressure, and such actions are quite unlike those of abrading or polishing.

Tomlinson established the following remarkable relation between friction and the elastic constants at ordinary temperatures:—

$$\frac{(\vartheta_A + \vartheta_B)^2}{\mu} \cdot 10^8 = \text{const. K.}$$

The bracket term involves rigidity and compressibility for the metals A and B used. The dimensions of the bracket term are the inverse of elasticity. This means

that friction is greatest for yielding materials such as lead and tin, and least for strong metals Fe and Ni, because the area of contact, other things being equal, would be greater for the softer materials.

Thus the coefficient of friction depends directly on the area of contact. Tomlinson observed values of μ for 10 materials in every combination, 55 results in all. The mean value found for K is 5·47, and 42 results are within 10 per cent. of this mean. This concurrence is all the more remarkable as the range in elasticity is so great—from $\cdot 3 \times 10^{11}$ for lead to $8 \cdot 9 \times 10^{11}$ for iron—and the range in melting-points from 200° C. to 1500° C.

The above equation, which we call Tomlinson's law, holds for the nine metals used and glass, and may prove universal. Accepting the law, the general principle follows that the elastic body-properties of these materials when polished hold good *right up to the outermost layer of molecules*. This is unexpected, for the surface of a solid is a peculiar region, normally possessing a large field of force, as shown by adsorption and other properties. Yet according to these experiments, coupled with the foregoing theory, the surface retains the full elasticity of the body of the solid and is for frictional purposes identical with it.

To account for the result we put forward the suggestion that the cleaning processes adopted by Tomlinson, in which he scours the surface with emery and then cleans and polishes, leaves that surface strained and hardened, and that the surface thus hardened is so well held by underlying layers that it possesses the full elastic strength of the whole solid mass. In contradistinction, our surfaces we take to be well annealed, and consequently softer and more frictional.

In the following Table are shown in the second column the values of μ after heat treatment for the unlike metals of the first column. In the third column are the mean values for the metals. Thus, for Ag–Fe we have 1·7, the mean of Ag–Ag (2·6), and of Fe–Fe (·8) (see foot of the Table). The fourth column shows how the actual values are, in all but two cases, below the mean values. In the last column are shown the affinities of the crossed metals for one another, as given by metallurgical data (see 'International Critical Tables,' vol. ii.).

It seems clear that affinity has an influence on friction, for when, as at the head of the Table, the metals have

small affinity the friction for the crossed metals is greatly reduced below the mean values, whereas, when further down the Table the affinity is great, the friction is less reduced, or in two cases, Cu–Ni, Al–Ni, is actually above the mean value.

These metallurgical data are obtained when temperatures are much higher than those of our experiments ; but we seem forced to admit from the Table that the affinities found at high temperatures hold good, in part, and affect our results even down to ordinary temperatures. By contrast with our results Tomlinson obtains results for unlike metals which are intermediate between those of the

Metals.	μ.	Mean values, μ_1.	$\mu-\mu_1$.	Affinities.
Ag–Fe	·3	1·7	—1·40	Two immiscible liquids, when melted.
Cu–Fe	·69	1·5	— ·80	
Ni–Fe	·69	·9	— ·20	
Ag–Ni	·86	2·1	—1·30	Two immiscible liquid solutions at Ni end.
Al–Fe	1·28	2·0	— ·72	One compound and solid solutions form.
Al–Cu	1·47	2·7	—1·23	Four ,,
Ag–Cu....	1·72	2·2	— ·48	No ,,
Cu–Ni	2·01	1·7	+ ·31	No ,,
Al–Ag	2·20	2·8	— ·60	Two ,,
Al–Ni	2·36	2·1	+ ·26	Three ,,
Fe–Fe	0·8			
Ni–Ni	1·1			
Cu–Cu	2·3			
Ag–Ag....	2·6			
Al–Al	3·1			

two individuals (*loc. cit.* p. 928) ; and so his values would seem to depend on elasticity chiefly, and slightly, or not at all, on affinity.

We will endeavour to account for this singularity in our results. We have already suggested that the strained surfaces of metals are annealed, and so are fundamentally changed by heat treatment. When in the annealed state we suppose that the natural affinities of the metals operate even at room temperatures. When the affinity is small the strength of the attachment of the surfaces is small and the friction is below that to be expected from the elastic moduli ; and when affinity is high the friction is thereby raised.

As an example take again the case of Ag-Fe. The friction of Ag–Ag is very great (2·6), of Fe–Fe small (·8), but the affinity of silver for silver and of iron for iron is greater than that of one metal for the other, alien, metal, and the resulting coefficient is only 0·3. It is well known that when two metals alloy with one another the distortion of the lattice structure and the consequent hardness of the alloy is greater, as the affinity between the metals is less. The affinity itself is small as the range of solubility is small. In pursuance of this idea the alloys formed at the head of the table would be harder than those further down, and if the alloying were deep enough in the lattice structure the hardening of the material would cause reduction in the area of contact and in μ, in accordance with Tomlinson's theory. But we do not consider this effect would influence results, since for metals in contact only a short time the alloying cannot be expected to be so deep-seated as to affect the general lattice strength.

If our views are correct, then as regards our experiments friction depends on the elastic moduli and affinity jointly.

Previous Researches.

Two investigators have adopted methods somewhat like ours :

1. C. Jacob (*Ann. d. Phys.* xxxviii. May, 1912) used a tilting method in a vacuum tube, and raised temperature by a heating coil outside it. Her results, so far as they bear on the present work, are :—

(*a*) Rise of temperature causes a small decrease in μ, with a minimum value between 150° and 200° C.

(*b*) At higher temperature there is a great increase in μ.

Definite values for μ are not given, but in both the above points there is general agreement with our results.

2. R. B. Dow (Phys. Rev. xxxiii. Feb. 1929) also used a tilting method. The metals used—brass, copper, zinc, aluminium—are enclosed in a vacuum into which various gases were introduced in turn. No heating coil was used,

so the results are not comparable with ours. The values of μ were all low, 0·1 to 0·5.

We wish to thank the National Physical Laboratory, and particularly Mr. S. L. Archbutt, of the Metallurgical Department, for kindly supplying pure metals and for advice in our work.

Summary.

1. Friction is performed in a vacuum chamber surrounded by a heating coil, so that the materials—aluminium, silver, copper, nickel, iron, and soda-glass—can be heated to 350° C. and exhausted. Every combination of the materials, like and unlike, is investigated. The tilt method of measuring friction is adopted.

2. After repeated heating and cooling in vacuum, the friction rises and finally settles to values which are, in general, practically constant between 15° and 350°. But for two or three cases there is a pronounced plus or minus temperature-coefficient of friction.

3. The coefficient of friction is much higher, especially for light loads, than when the materials are treated in open air. The suggestion is put forward that, for the metals, the heat treatment causes thorough surface relaxation of the material, and the surface is thus not only softened but is radically different in character from one ground and polished. The ground and polished surface possesses the elastic characteristics of the general body of the metal; the relaxed surface is not only softer, but is more free to assert its natural affinity for other metals. The result of the heat treatment is that friction is increased by increase of softness and increase of affinity.

4. As the load on the surface is increased up to about 180 gm. μ decreases, but reaches limiting values which are, even then, considerably higher than those for fully strained surfaces.

5. It is clear that the term "coefficient of friction" has to be defined in terms of the history and physical condition of the reacting surfaces.

LXXV. *A Note on the Representation of Crystal Structure by Fourier Series.* *By* Prof. W. L. Bragg, *F.R.S.*, *Langworthy Professor of Physics, Manchester University, and* J. West, *Moseley Research Student (Royal Society)* *.

1. *Introduction.*

In a former paper [1] an account has been given of the use of a double Fourier series to represent the projection of a crystal structure on a given plane. If measurements of X-ray diffraction are made for all planes round a given zone in a crystal, and certain further information about "phase" is available, a projection can be made by summing a Fourier series with two variables. It was shown that such projections give a vivid picture of the atomic arrangement in the crystal.

In the present paper we wish to discuss the extent to which the representation is a faithful image of the actual crystal structure. It will be seen that the term "image" is appropriate, for it suffers from defects precisely analogous to those caused by diffraction in an optical image. It is interesting to see how close the analogy is and how constants of X-ray analysis can be defined which have exactly the same significance as the "resolving power" and "numerical aperture" of a microscope.

2. *The Fourier Representation.*

The use of a Fourier series to represent the results of X-ray analysis was first suggested by W. H. Bragg [2], and developed by Duane [3], Compton [5], and Havighurst [4]. The method employed in the previous paper in the 'Proceedings of the Royal Society' was based on the same principle as their work, but is more extended in its scope. If the crystal structure, when projected parallel to a zone on any given plane, has a two-dimensional unit cell of area A and sides b and c, then the density of scattering matter (number of electrons per unit area) at a point y, z referred to the cell sides as axes is given by

$$\rho(y,z) = (1/\mathrm{A}) \cdot \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \mathrm{F}(okl) \cos 2\pi(ky/b + lz/c).$$

* Communicated by the Authors.

F (*okl*) measures the reflecting power of the plane (*okl*) as previously defined, and the projection in this case is supposed to be parallel to the *a* axis of the crystal.

If the series is calculated for points distributed throughout the unit cell the atoms appear as concentrations of greater density. Though they sometimes overlap each other in the projection, certain of them often stand out clear of their neighbours. In such cases it is possible to sum

Table I.

$\sin\theta/\lambda$.	F values. Room-temperature. Na+.	Cl−.	F values. Atoms at rest. Na+,	Cl−.
0	10·0	18·0	10·0	18·0
0·05	9·82	17·05	9·86	17·12
·10	9·35	15·02	9·50	15·23
·15	8·61	12·80	8·93	13·18
·20	7·71	10·93	8·21	11·50
·25	6·76	9·48	7·45	10·24
·30	5·81	8·44	6·67	9·30
·35	4·90	7·37	5·93	8·60
·40	4·10	6·58	5·25	8·05
·45	3·29	5·83	4·51	7·63
·50	2·76	5·30	4·07	7·24
·60	1·83	4·14	3·22	6·49
·70	1·23	3·13	2·63	5·76
·80	0·83	2·28	2·23	5·06
·90	0·57	1·62	1·97	4·42
1·00	0·37	1·10	1·73	3·84
1·1	0·25	0·74	1·59	3·34

up the total amount of scattering matter, and so to obtain a direct count of the number of electrons in the atom.

There are many problems of crystal structure which it would be interesting to solve in this way. Havighurst and Compton have used Fourier series with a single variable to test such questions as the ionization of sodium and chlorine in sodium chloride. The Fourier series with two variables is on the face of it a more powerful method, because it separates the atoms more from each other. Further, the series is just as easy to manipulate for large

unit cells containing many atoms as for small cells of a simple type such as that of sodium chloride. A cell of large dimensions yields measurements of scattering at small glancing angles, and these are essential in finding out the positions of the outer electrons in the atoms. An instance of a problem on which light may be cast by the Fourier representation is the distribution of electrons between silicon and oxygen in silicates.

The difficulty of such investigations arises partly from errors in the experimental measurement of diffraction (values of F), and partly from the termination of the Fourier series before its coefficients have become very small.

TABLE II.

Values of F(*okl*) for Crystal at Room-temperature.

	0.	1.	2.	3.	4.	5.	6.	7.	8.
									$\rightarrow k$
0	28·00	—	16·14	—	7·94	—	3·64	—	1·40
1	—	4·09	—	2·44	—	2·15	—	1·12	—
2	19·90	—	13·94	—	7·31	—	3·39	—	1·32
3	—	2·47	—	2·55	—	1·94	—	1·00	—
4	12·15	—	9·56	—	5·73	—	2·75	—	1·14
5	—	2·58	—	2·32	—	1·57	—	0·78	
6	7·32	—	6·19	—	3·93	—	1·97	—	
7	—	2·16	—	1·78	—	1·11	—	0·55	
8	4·21	—	3·64	—	2·42	—	1·27		
9	—	1·40	—	1·11	—	0·70			
10	2·29	—	1·95	—	1·34				
11	—	0·76	—	0·62					
$l\downarrow$ 12	1·14	—	1·00						

The series is terminated because it is impossible to measure more than a certain number of reflexions with a given X-ray wave-length, and its effect is precisely like that which limits the resolving power of an optical instrument. In the present paper we wish to examine the question of resolving power alone, and will take an ideal case where we suppose no experimental errors to have been made.

3. *Diffraction Effects in the Image.*

It has been shown by James Waller and Hartree [6] that the observed X-ray diffraction by sodium chloride corresponds very closely with that to be expected from the

Hartree model of atomic structure. We have therefore tested the Fourier representation by taking an ideal crystal of sodium chloride composed of such Hartree atom-models, calculating its diffraction as if observations were being made experimentally for a limited range of crystal planes, expressing these results in the form of a double Fourier series, and then seeing with what faithfulness the projection represents the original model *.

The most convenient projection which separates the sodium and chlorine atoms is that upon the plane (110);

Table III.

Values of F(*okl*) for Crystal when Atoms are at rest.

	0.	1.	2.	3.	4.	5.	6.	7.	8.
0	28·00	—	17·58	—	11·23	—	7·76	—	5·53
1	—	4·12	—	2·77	—	3·24	—	2·51	—
2	20·67	—	15·76	—	10·74	—	7·53	—	5·44
3	—	2·64	—	3·05	—	3·16	—	2·39	—
4	14·38	—	12·52	—	9·49	—	6·94	—	5·12
5	—	3·04	—	3·28	—	2·93	—	2·15	
6	10·74	—	9·87	—	7·99	—	6·18	—	
7	—	3·24	—	3·05	—	2·51	—	1·81	
8	8·25	—	7·76	—	6·63	—	5·34		
9	—	2·79	—	2·51	—	2·06			
10	6·49	—	6·18	—	5·44				
11	—	2·16	—	1·94					
12	5·11	—	4·93						

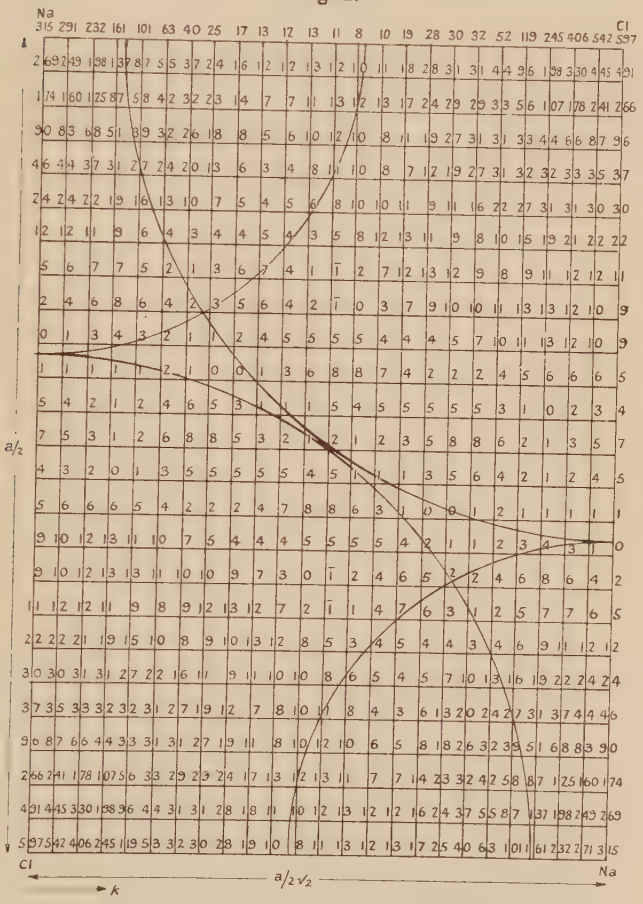
($l\downarrow$, $\rightarrow k$)

the unit cell then has the dimensions a by $a/\sqrt{2}$, and contains two atoms of each kind. It is only necessary to represent one quarter of such a cell in order to see the pattern.

* Havighurst (Phys. Rev. xxix. p. 1, 1927) has made a precisely similar examination with a single Fourier series and a model of the sodium atom, and comes to the conclusion that "the value of the retention of the temperature factor in the experimental F-curves as a means of securing a rapidly convergent series and therefore of obtaining a more trustworthy Fourier analysis is clearly demonstrated." As will be seen, we arrive at the same conclusion about a temperature factor for the double Fourier series. It is interesting to examine this series in detail, however, because it is so widely applicable to complex crystals, where the single Fourier series is helpless, and, further, because the optical analogies are so striking.

We have supposed the measurements to be made up to a glancing angle θ_0 equal to $\pi/6$ ($\sin\theta_0 = 0{\cdot}5$), and for a wave-length of 0·615 Å. ($Rh\,K_\alpha$ radiation). The series has been calculated for values of F at room-temperature (290° absolute) and for an ideal crystal of atom-models

Fig. 1.



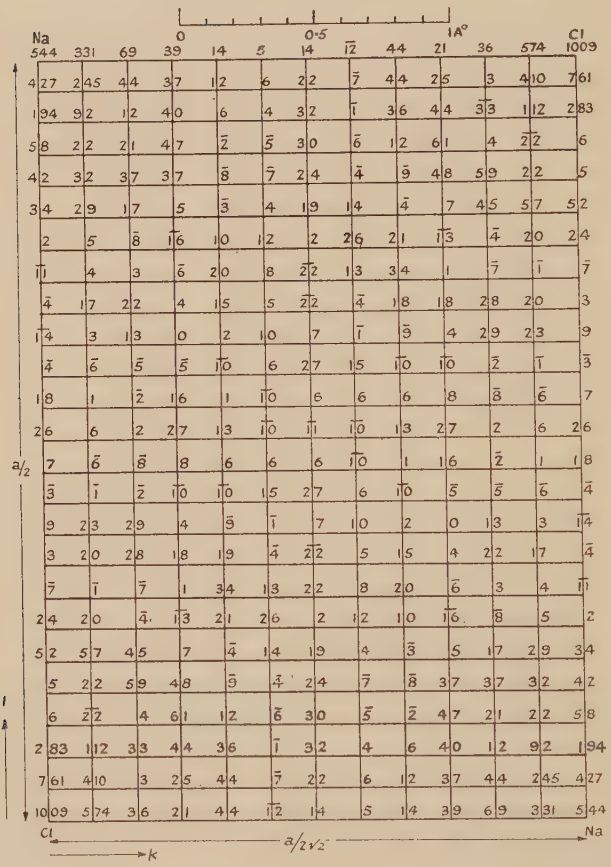
Projection on (100) [*i.e.*, (110) true cell of side $a = 5{\cdot}628$ Å].
Crystal at room-temperature.

unaffected by any movement or by the so-called zero-point energy. The F values adopted for each atom in the two cases are given in Table I.; they are taken or deduced from those given by James, Waller, and Hartree.

The F values for the unit cell used in the projection and appropriate to each crystal plane are given in Tables II. and III.* These values are normally found by experiment. Planes (*okl*) in our adopted unit cell correspond to planes (*kkl*) in the *true* unit cell of rock-salt.

In the case of the crystal at room-temperature the

Fig. 2.



Projection on (100). Crystal with atoms at rest. The number of points taken here is less than in fig. 1, but a more detailed analysis is given in figs. 4 and 5.

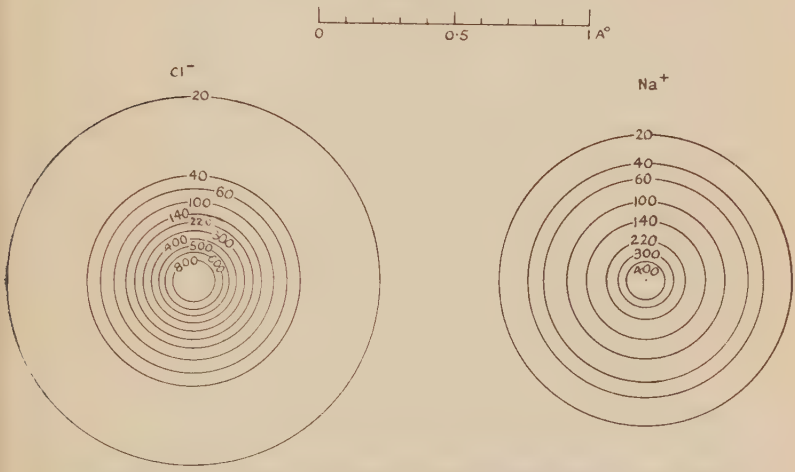
* The numbers given in Tables II. and III. in the figs. 1–5 should strictly be doubled, since there are two molecules in the unit cell adopted. It is necessary to note this when estimating the distribution of scattering matter in the diagrams concerned.

calculated electron density throughout the unit cell is shown in fig. 1. The numbers represent the summation of the Fourier series

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} F(okl)\cos 2\pi(ky/b + lz/c),$$

and, after being doubled (see note above), must be divided by the area of the cell (22·40 Å.) in order to get the actual density. It is convenient to use the Ångström as a unit of length in such a projection. Although the coefficients of the series are still appreciable when the series terminates, they are small compared with the initial coefficients.

Fig. 3.



Projection on plane of atom-models used in calculations. The scale is the same as that for figs. 1 and 2. The numbers on all three diagrams have the same meaning.

The corresponding projection for the ideal crystal composed of atoms unaffected by movement is shown in fig. 2. In fig. 3 we give to the same scale as the two previous diagrams the projections on a diametral plane of the two kinds of atom-models. The contour lines in all three diagrams are drawn through points of corresponding density.

4. *Discussion of Fig.* 1.

In regions widely removed from the centres of the atoms the density is very small, and it nowhere has an

appreciable negative value. Although sodium and chlorine overlap in the direction of the short axis (perpendicular to the (110) plane in the normal unit cell) they are fairly well separated elsewhere. The image reproduces correctly the spherical symmetry of the atom-models, and an attempt can be made to separate the one atom from the other, apportioning the density between the two atoms in the part where they overlap, so that the components are in accord with the values found elsewhere at corresponding distances. The result of an electron count gave 17·5 for chlorine and 10·5 for sodium, and is close to the values 18 and 10 for the ionized atoms.

The count does not, however, decide definitely the question as to whether the atoms are ionized or not, although the figures on which the projection is based were appropriate to ionized atoms. At first sight this may seem to be due to an error in drawing the boundary between the two atoms. Such an error could arise from the presence of irregularities in the projection—discussed below as "diffraction effects"—caused by terminating the Fourier series whilst the coefficients are still appreciable. These irregularities, as we shall see later, can seriously affect an electron count, and although in fig. 1, by making use of a temperature factor, they are by no means so marked as in fig. 2, they are still present.

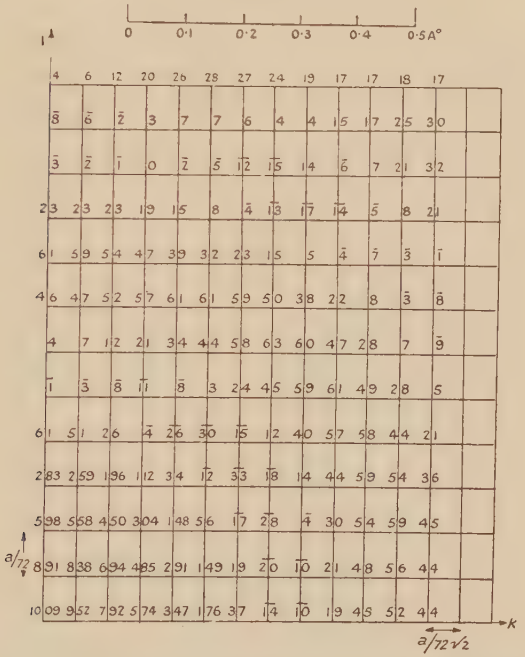
A consideration of Table IV. will, however, show that the defect is more fundamental, and that the count would still be incorrect even if the irregularities just mentioned were removed by the use of a higher temperature coefficient. The boundaries chosen for the two atoms are indicated by the arcs in fig. 1, which have radii 1·73 Å. for Cl^- and 1·08 Å. for Na^+. Table IV. shows that whereas the latter radius includes in effect all the electrons of Na^+, the former only includes about 17·4 electrons of Cl^-. The remainder falls within the boundary of the sodium atom, and being automatically included in the count for Na^+ in the projection, is wrongly ascribed to it. This seems to suggest that a correct electron count is not possible. However, we have to remember that the original F curves used were calculated for free ions, and since presumably F values measured experimentally will correspond to ions whose electrons are compressed into a smaller region by the mutual approach of the atoms, it would seem possible that a correct count might be made

in an actual experiment provided the difficulties of obtaining accurate F values are successfully overcome. At the same time it is clear that considerable caution should be exercised in interpreting the results and that rigorous conditions must be satisfied before they can be relied upon.

5. *Discussion of Fig.* 2.

In this case the numbers show oscillations between positive and negative values all over the unit cell. The

Fig. 4.

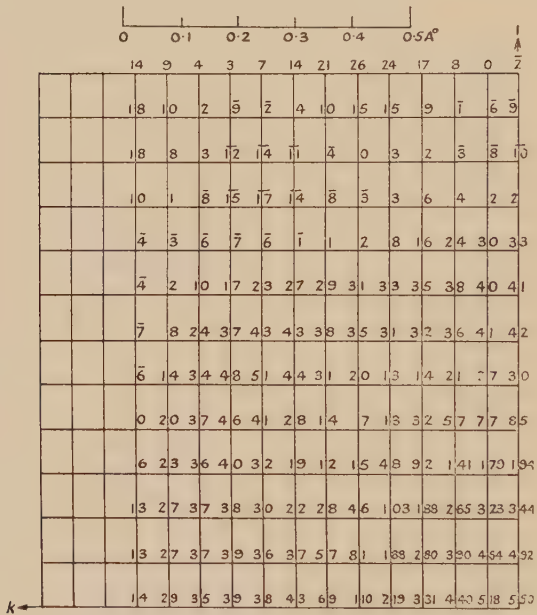


Portion of fig. 2 (neighbourhood of *chlorine* atom) magnified and calculated for closer points.

centres of atoms are marked by higher peaks than in fig. 1 but they are surrounded by concentric rings of positive and negative density which, by overlapping, produce the complicated aspect of the projection. *These irregularities have no counterpart in the atomic structure.* They are due to the termination of the Fourier series whilst the coefficients are still large, and are precisely

analogous to the diffraction effects produced by an optical instrument. The atomic model has a large concentration of scattering matter at its centre, and an effect is produced like the diffraction rings around the image of a luminous point. In the present case it is *amplitude* and not *intensity* which is plotted, so that the diffraction rings give both positive and negative contributions and are more important as compared with the central maxi-

Fig. 5.



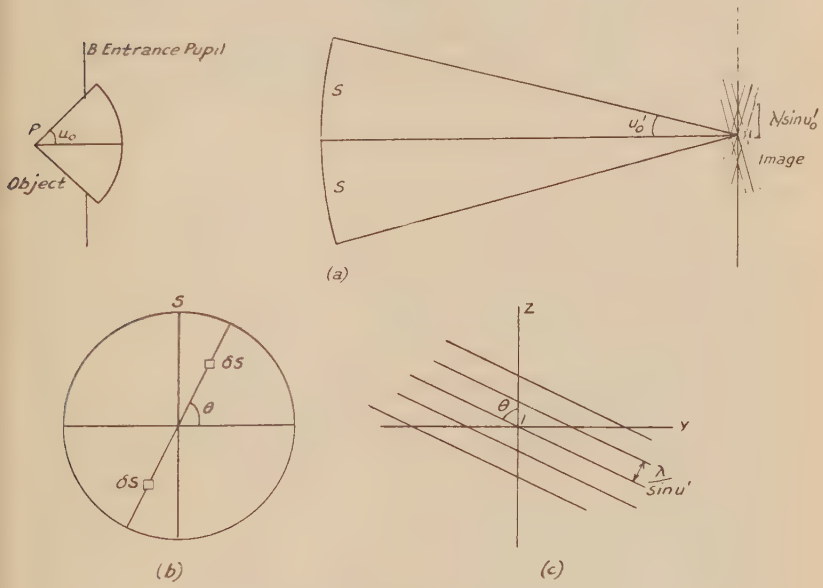
Portion of fig. 2 (neighbourhood of *sodium* atom) magnified and calculated for closer points.

mum. A negative value for the density has no physical significance in this case, and its existence shows that the picture is artificial.

In figs. 4 and 5 the electron density around atoms of either kind is shown plotted on a larger scale and calculated for closer points. Electron counting is much more difficult here than in fig. 1. If, for instance, we count merely the number of electrons in the central region of positive density for chlorine we obtain 11·3, whereas we

should have 18. Clearly the scattering matter represented by the outer rings must be included in the count, but it is difficult to know where to draw the boundary between the atoms, since the oscillations overlap in all directions. The precise point at which counting stops influences considerably the total number obtained, whereas in fig. 1 it matters little where the boundary is placed, since the region between the atoms has a vanishing density.

Fig. 6.



6. *The Optical Analogy.*

The origin of these circular bands of positive and negative density is clear if we treat the projection as analogous to an image formed by an optical instrument with a circular stop.

Consider the image I of a luminous point P (fig. 6). The rays from P which traverse the instrument are limited by a circular stop at some point in the system which is equivalent in its effect to the " entrance pupil " B. After traversing the system a wave coming originally from P converges to build up the image at I. The angular aperture is measured by $2u_0$ and the angle of projection by $2u_0'$.

The customary treatment of diffraction divides a wave front such as S into elements, the secondary waves from which combine to form the image at I. A pair of elements of area δS symmetrically placed at opposite sides of the wave centre contributes an amplitude proportional to

$$\delta S \cos(2\pi t/T) \cos 2\pi(\cos\theta \sin u' \,.\, y/\lambda + \sin\theta \sin u' \,.\, z/\lambda),$$

where $2u'$ is the angle the elements subtend at I. The elements and the wave system they produce are shown in fig. 6 (*b* and *c*). The resultant amplitude of vibration for a given point *yz* in the image plane is got by integrating this expression over the whole wave surface. The pattern has circular symmetry, the amplitude of vibration at a distance *r* from the centre of the image being given by

$$\text{constant} \,.\, J_1(2m)/m$$
$$= \text{constant}\left(1 - \frac{1}{2}\left(\frac{m}{1}\right)^2 + \frac{1}{3}\left(\frac{m^2}{2}\right)^2 - \frac{1}{4}\left(\frac{m^3}{3.2}\right)^2 + \ldots,\right.$$

where $m/\pi = r(\sin u_0')/\lambda$.

The square of this expression yields the familiar diffraction pattern for a circular aperture, but we are here concerned with amplitude and not intensity.

The curve for the amplitude $J_1(2m)/m$ is plotted in fig. 7. Its maxima, minima, and points where it crosses the axis occur as follows :—

	Magnitude.	m/π.
Central maximum	1·0	0
Zero value	—	0·61
Minimum (first diffraction ring)	0·132	0·81
Zero value	—	1·116
Second maximum (second diffraction ring)	0·064	1·333
Zero value	—	1·619

In the same figure the amplitude for a slit whose width is equal to the diameter of the stop is shown for comparison, the expression in this case being

$$(\sin 2m)/2m.$$

Each pair of equal and opposite elements in the wave converging to form the image builds up in the image plane a contribution to the amplitude like that of a set of standing waves. Elements close to the centre give standing waves of long wave-length, whilst those farther apart give waves of short wave-length because the wavelets from those elements cross at a greater angle. The

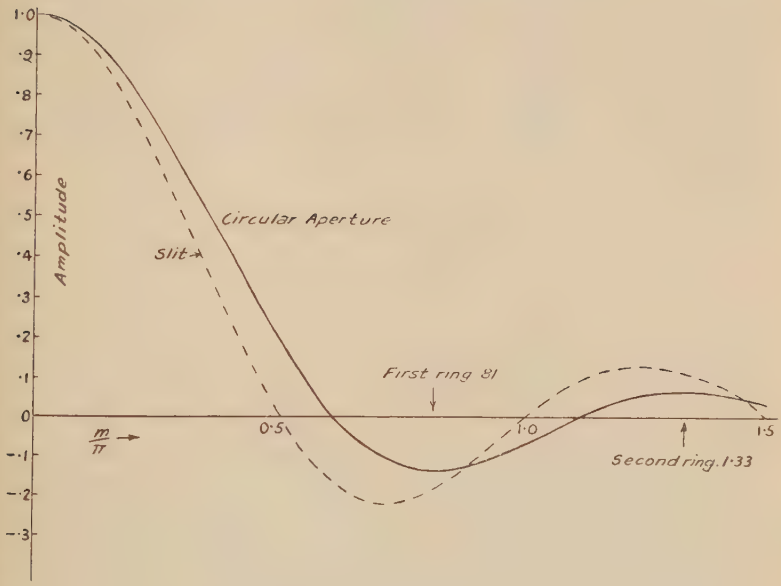
shortest wave-lengths which go to compose the image have a length $\lambda/\sin u_0'$, as shown in the figure. This corresponds to a distance in the object of

$$\lambda/n \sin u_0 = \lambda/a,$$

where a is the *numerical aperture* of the instrument, n being the refractive index of the medium in which the object is placed.

To sum up, the diffraction image at I is built up by integrating these standing waves, which cross each other

Fig. 7.



Optical diffraction rings for circular aperture and slit.

in all directions and have wave-lengths varying from infinitely great to a minimum value corresponding to a distance λ/a in the object. These waves are all in phase at the centre of the image where the effect is greatest.

Suppose now that the projection of a crystal structure is being found which for simplicity we take to consist of a simple network of single electrons. The image of such a structure is given by the Fourier series

$$\rho(y, z) = (1/\mathrm{A}) \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \mathrm{F}(okl) \cos 2\pi(ky/b + lz/c),$$

in which all values of F are unity. All spectra are measured for planes whose spacings are not less than a given amount d_0, thus imposing limitations on possible values of k and l. The smaller d_0, and the larger the dimensions of the unit cell, the greater will be the number of spectra. It is easy by considering the reciprocal net to show that the number of spectra approximates to

$$\pi A/d_0^2.$$

The image of each electron is formed by sets of cosine waves crossing in all directions, as in the optical case, only now their number is finite, and we are dealing with a Fourier series instead of the integral over the wave surface in the optical case. Nevertheless, when the number of spectra is very large, the form of the image will be like that due to optical diffraction. The electron density at each point in the image at a distance r from a corner of the net is given by

$$\rho(r) = \frac{1}{A} \cdot \frac{\pi A}{d_0^2} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \cos 2\pi(ky/b + lz/c)$$

$$\simeq \frac{\pi}{d_0^2} \cdot J_1(2m)/m,$$

where $m/\pi = r/d_0$.

The size and shape of the unit cell of the network do not affect this last expression. When the number of spectra is very large, the image of each net point approximates to what it would be if it were the only point in the field of view; expressed otherwise, the diffraction patterns are only slightly affected by overlapping. It is as if we were looking through the microscope at a regular two-dimensional array of luminous points producing cross spectra in the region S (fig. 6). Waves from these spectra recombine in the image plane to reproduce an array of images, each of which is surrounded by diffraction rings, much as it would be if it were the only point in the field.

Now, the shortest wave which goes to build up the image of our network corresponds to the shortest distance d_0 between reflecting planes. If all spectra are measured up to a glancing angle θ_0, then

$$d_0 = \lambda/2 \sin \theta_0.$$

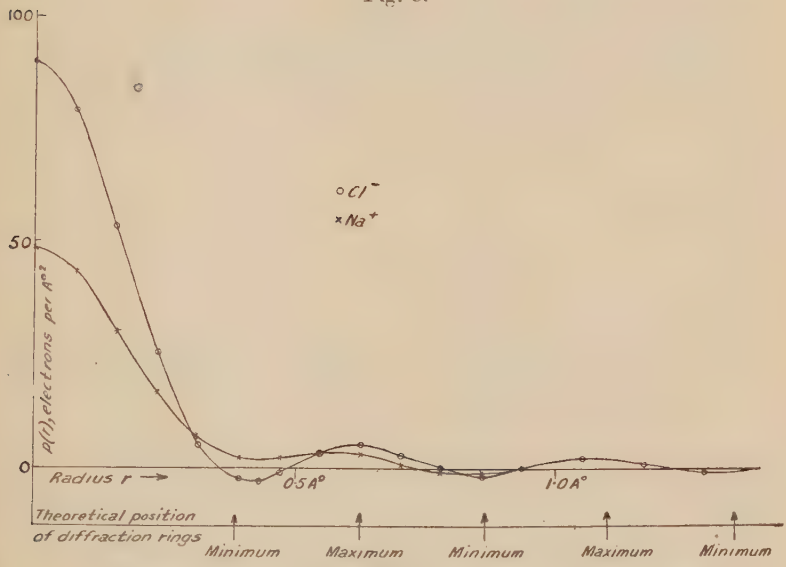
In the optical image the corresponding distance is λ/a. Thus, *the numerical aperture in X-ray analysis must be*

defined as $2\sin\theta_0$. In the microscope points cannot be resolved unless their distance apart is greater than $0{\cdot}6\ \lambda/a$. Similarly in X-ray analysis detail cannot be distinguished unless it is on a coarser scale than $0{\cdot}6\ d_0$ or $0{\cdot}6\ \lambda/2\sin\theta_0$.

7. *Diffraction Effects in the Fourier Projections.*

In the Hartree atom-models there is a strong concentration of scattering matter at the centre of the atom. This gives rise to diffraction rings on the projection like

Fig. 8.



Diffraction rings of Cl^- and Na^+ compared with optical rings.

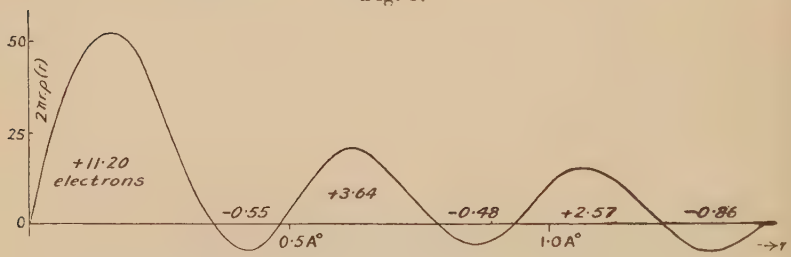
those due to a bright point at the atom centre. The first diffraction ring has its greatest negative value at $0{\cdot}81\ d_0$, the next its greatest positive value at $1{\cdot}33\ d_0$, and so forth. In the present case the maximum value of θ is $\pi/6$, so that the "angular aperture," $2\sin\theta_0$, is unity, and d_0 is equal to λ, *i. e.*, 0·615 Å.

In fig. 8, $\rho(r)$ is plotted against r for sodium and chlorine, using the data of figs. 2, 4, and 5, and disentangling as far as possible the overlapping system of rings. The corresponding positions of the maxima and minima due to the diffraction rings of a luminous point are marked in the

diagram. It is obvious that the rings of positive and negative density in the projection correspond precisely to optical diffraction rings. They have no counterpart in the atomic structure, and are merely due to the termination of the Fourier series whilst the coefficients are still large. The image of either atom consists of a faithful representation of the more diffused outer parts, together with a system of diffraction rings, due to the large concentration at the centre, superimposed upon it.

By plotting $\rho(r).2\pi r$ against r (fig. 9) it can be seen why the electron count for the central diffraction maximum is so small. Owing to their larger radius the outer rings contribute considerably to the total number of electrons. The contributions of the successive rings to the total number of electrons in the chlorine atoms are

Fig. 9.



Distribution of electrons in projection of Cl^- derived from fig. 4.

shown in fig. 9. The central maximum gives 11·2, a sum which is raised to 15·2 by the rings shown in the figure. The count is not yet complete, but beyond this point it is impossible to separate the diffraction effects of neighbouring stoms.

The concentration of scattering matter at the centres of the sodium and chlorine atom-models, which causes these diffraction effects, is shown in Table IV. The Hartree model atoms have been projected on a diametral plane. The figures in the Table give the radii of circles within which successive integral numbers of electrons are contained in the projection. Fig. 3 was based on this table.

8. *Conclusions.*

The use of a Fourier series which is incomplete, in that the coefficients are still large when it is terminated, leads

to an image of the crystal structure which has defects like those produced by optical diffraction. Such defects are strongly marked in fig. 2, whereas in fig. 1 they are much less apparent. The coefficients fall to a much smaller value in this latter case, because a "temperature factor" $e^{-B\sin^2\theta}$ has been added to allow for thermal agitation. Although this agitation will increase the overlapping of the atoms to a certain extent, the effect is more than

TABLE IV.

Cl−.		Na+.	
Radius of circle.	Number of electrons within circle.	Radius of circle.	Number of electrons within circle.
3·19 A.	17·96	1·06 A.	9·96
2·40	17·8	0·575	9
1·55	17	0·446	8
1·14	16	0·367	7
0·936	15	0·298	6
0·772	14	0·239	5
0·692	13	0·186	4
0·522	12	0·128	3
0·440	11	0·074	2
0·314	10	0·027	1
0·256	9	0	0
0·213	8		
0·181	7		
0·149	6		
0·117	5		
0·096	4		
0·064	3		
0·043	2		
0·027	1		
0	0		

compensated for by the disappearance of the diffraction effects, and the line of demarcation between the atoms is actually much more evident. An electron count in fig. 1 can be made with considerable accuracy, whereas in fig. 2 it is impossible.

This suggests that it would be desirable in all cases to multiply the coefficients by an arbitrary factor $e^{-B\sin^2\theta}$, choosing a value of B which makes the final coefficients very small. If, for instance, the maximum value of

$\sin\theta_0$ is 0·5, a value of 2·3 for B gives the following factors:—

$\sin\theta$.	$e^{-B\sin^2\theta}$.
0·1	0·912
0·2	0·692
0·3	0·437
0·4	0·229
0·5	0·100

It is somewhat surprising to see to what a small value of the average amplitude of thermal agitation this value of B corresponds. The factor in terms of the average mean square of the amplitude, $\overline{u_x^2}$, is

$$e^{-\frac{8\pi^2\sin^2\theta}{\lambda^2}\cdot\overline{u_x^2}}.$$

For $\lambda=0{\cdot}615$ Å, and $\frac{8\pi^2\overline{u_x^2}}{\lambda^2}=2{\cdot}3$ as above, we have

$$\overline{u_x^2}=(0{\cdot}21\text{ Å.})^2.$$

Such an amplitude hardly affects the extent to which the atoms overlap. The Fourier series will give a very faithful picture of the actual crystal structure with this assumed thermal agitation. Electron counting should therefore be more reliable, and one can be sure that no false detail due to diffraction will appear.

Compton ('X-rays and Electrons,' p. 164) develops a formula for the amount of scattering matter $U_r dr$ between radii r and $r+dr$ in the atom

$$U_r dr=\frac{8\pi r}{D^2}\sum_1^\infty nF_n\sin\left(\frac{2\pi nr}{D}\right)dr,$$

where each F value is measured or deduced for an order of reflexion n from a simple crystal of spacing D composed of the one kind of atom. This series converges very slowly. Using experimental determinations of F, Compton obtained a curve for chlorine, for instance, with a series of "humps." By comparing the curve with the Hartree model (with which we know the experimental measurements to be in accord) it is clear that the humps are similar to the diffraction effects treated above, and do not correspond to the actual distribution in the chlorine atom. Havighurst used the same formula (Phys. Rev. *loc. cit.*)

and pointed out the importance of a temperature coefficient in making the series converge. Even in Havighurst's results for chlorine, however, humps appear which have no counterpart in the Hartree model, and, indeed, Havighurst expresses doubts of their reality. It seems clear that we can only trust details of structure indicated by the Fourier analysis when the actual or applied temperature coefficient causes the series to converge so fast that the last terms are vanishingly small. Unless this is the case, many real-looking features appearing in the representation must be distrusted.

By making the final coefficients of the Fourier series vanishingly small in this artificial way we are sacrificing a part of the information about the crystal structure which the experimental determinations have provided. A comparison of figs. 1 and 2 shows what is being lost. The higher terms define more sharply the positions of the atomic centres by making the peaks in the density higher and more concentrated. They give, in fact, more exact information about the positions of the atomic centres in more complicated crystals where these are not fixed by symmetry considerations. The electron count, on the other hand, is mainly determined by the earlier terms in the series. The higher terms " trim " the crude picture of the structure given by the earlier terms, and in order to make this trimming effective and to cut out false detail it is quite justifiable to multiply the coefficients by an arbitary temperature factor. The introduction in an optical instrument of a screen with a graded absorption increasing towards the edge would cut out false detail due to diffraction in just the same way.

In a further paper we hope to apply these results to experimental measurements on crystal diffraction.

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Manchester University,
July 28, 1930.

LXXVI. *The Conduction of Electricity in Liquid Dielectrics.* *By* D. H. Black, *Ph.D.*, *and* R. H. Nisbet, *B.Sc.* (*International Telephone and Telegraph Laboratories, Incorporated*) *.

[Plates VI. & VII.

ALTHOUGH a considerable amount of work has been carried out in recent years on the electrical properties of solid insulating materials, there do not appear to have been many investigations of any great value published on the phenomena associated with liquid dielectrics. As is known to all who have attempted the subject, the experimental difficulties encountered when dealing with such liquids are very great, being greater in many respects than those associated with solid dielectrics. Consequently the experimental data are somewhat meagre, and it has not been possible to draw any very conclusive deductions from them, although it seems to be generally assumed that the direct current conductivity of liquid dielectrics is similar in nature to the electrical conductivity of ionized gases. It is therefore the more regrettable that some writers, while making no attempt to explain some of the more well-known experimental facts, do not seem to appreciate fully some of the consequences to which the theory of conduction in gases leads.

Such well-known facts as the decrease in current with time and variations of resistance with voltage are often attributed to the action of counter E.M.F.'s formed by the accumulation of space-charges; but no attempt is made to account for the absence of the residual discharge currents which these space-charges would seem to entail. Largely on account of the absence of such discharge currents one of us [1] put forward the hypothesis that the formation of a contact resistance at one or both of the electrodes was responsible for the observed facts; and the simple mathematical relationships derived were found to agree with the bulk of the experimental data obtained up to that time. In this paper we give the results of further experiments on direct current conduction in insulating oils, and discuss their bearing on the nature of the processes involved.

* Communicated by the Authors.

Experimental Methods.

Measurements on the steady value of the resistance of oil films have been extended to higher and to lower potential gradients. The same testing vessel as mentioned in the previous paper has been used throughout, the electrode area being 240 sq. cm., and the same attention has been paid to the control of temperature. When possible the resistances were measured with a moving-coil galvanometer in conjunction with a standard megohm, and when the currents were too small for this method a null electrometer system of measurement was used [2], [3].

An Einthoven string galvanometer has been used for taking measurements at short time intervals after the application, reversal, or removal of the potential. The deflexions were recorded on a moving bromide paper strip, and time intervals were marked on the record by means of a toothed wheel which interrupted the beam of light, the wheel being controlled by an electrically driven tuning-fork. Owing to the progressive tightening of the galvanometer fibre with increasing deflexion the readings were not proportional to the current, and it was therefore necessary to take calibration deflexions for every observation. These calibrations were carried out by applying the potential across the megohm, and recording the deflexions due to the resulting current at three or more settings of a universal shunt connected across the galvanometer. On account of the large capacity currents involved, the galvanometer was shunted by a low resistance when the applied potential was changed, and a small fraction of a second later the shunt was disconnected and the full current allowed to flow through the galvanometer. The value of the shunt resistance was so chosen that the flow of the capacity current produced a small kick in the galvanometer, thus indicating the exact moment at which the change in the potential took place. When recording the current flowing through an oil it was necessary to short-circuit the megohm, as, if it was in series with the film, its presence was found to alter the shape of the curve, the effect probably being due to inductance since the resistance was wire wound.

The liquids used in the experiments quoted in the paper were oils of the paraffin class; and for the majority of the tests two grades of liquid paraffin were used.

Liquid paraffin "A" was a highly purified medicinal paraffin, and liquid paraffin "B" an oil supplied as extra high viscosity liquid paraffin. Both these oils were supplied in sealed glass bottles and received no further treatment other than de-gassing under vacuum for several hours. In the experiments with the special apparatus for the detection of contact resistances, which is described later, a light mineral oil supplied for turbine lubrication was employed.

Steady Current Phenomena.

Many writers on the subject have assumed that conduction of electricity in liquid dielectrics is very similar in nature to conduction in ionized gases, and in many instances they attribute the decrease in the current with time * to the gradual formation of space-charges in the liquids †. In the case of solid dielectrics also the accumulation of charges in the material is generally assumed to be the cause of the decrease in current with time; and in this case it is assumed that the disappearance of such charges on removal of the applied potential gives rise to the observed residual discharge currents. In the case of liquid dielectrics, however, residual discharge currents of appreciable magnitude are seldom observed, and therefore it did not seem reasonable to attempt to explain the decreases in current by the same mechanism for both solids and liquids. It was mainly on account of the absence of such discharge currents in oils that the attempt was made to explain the observed effects in some other manner, namely, by the formation of contact resistances.

In the previous paper a simple expression was obtained connecting the final resistance, R_s, of an oil film with the final value of the current, I, flowing across it. This was

$$R = R_0 + nI, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

n being a constant at any particular temperature, while R_0, which was considered to be the true resistance of the

* On account of the apparent similarity to the phenomena observed with solid dielectrics this decrease is sometimes referred to as an "absorption current," but we do not consider the term satisfactory when referring to liquids.

† A decrease in current with time would presumably take place in gases, but the final steady conditions would be attained so rapidly that such a charge has not yet been recorded.

oil, was proportional to the film thickness. This can be written in the form

$$V = R_0 I + nI^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where V is the applied potential, and is then identical with that obtained for the relationship between current and voltage for conduction in gases under certain conditions, more particularly for conduction in flames [4], [5]. According to the gas theory, from which the above formula can be derived, there are sudden drops of potential at the electrodes, and the distribution of ions in the central region is the same as that existing before the application of the potential. Since the current flowing is the same throughout the space between the electrodes, these drops of potential can only occur in regions where the specific resistance is higher than in the body of the liquid. Deficits in the numbers of both kinds of ions in these regions are the cause of the increased resistance, and the space-charges arise through there being fewer positive than negative ions near the positive electrode, and fewer negative than positive near the negative electrode.

In the theory of contact resistances previously outlined the form in which these resistances existed was not stipulated *; and such a general resistance theory may perhaps be regarded as including the gas theory with its concentration of resistances at the electrodes. It is, however, difficult to trace any connexion between the methods by which the results are obtained in the two cases.

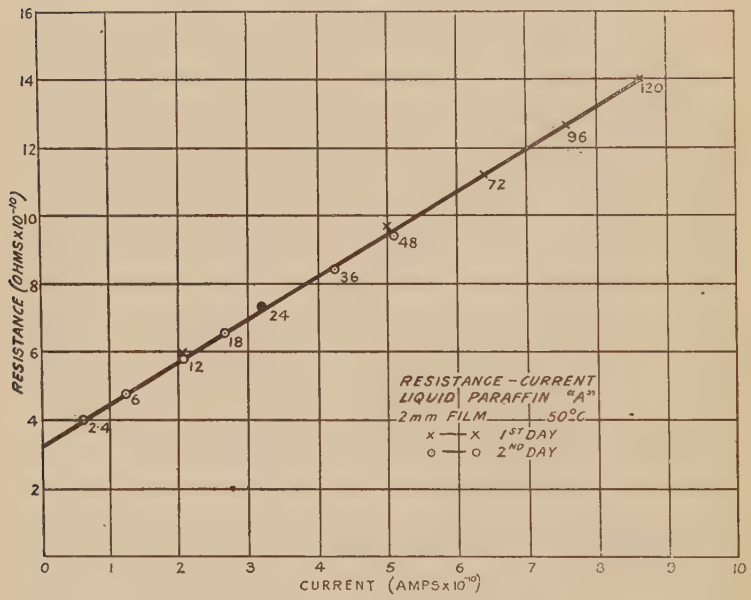
It seems to have been assumed by some writers † on this subject that under low stresses the resistance of a film of liquid dielectric remains constant as the potential is increased. Curtis mentioning a value of 300 volts/cm. as the upper limit to the range over which Ohm's law is found to hold. However, the theories discussed above require that there should be a linear relationship between resistance and current even for small values of the current, and our experimental results definitely support

* It was suggested that these resistances might possibly be due to the formation of gas films on the electrodes, but it has since been found that the experimental facts upon which this suggestion was based arose through the presence of occluded gases in the oils, and were not directly due to the passage of the current.

† See, for example, Curtis [6] and Nikuradse [7].

this conclusion. We have extended the measurement of resistance to stresses lower than those mentioned in the previous paper, and one set of results is shown in fig. 1, where the linear relationship is seen to hold within very close limits *. The actual voltage applied at each point is indicated on the graph, and the values cover a variation in stress of from 600 to 12 volts/cm. Incidentally it may be mentioned that in some instances steady conditions were not attained until five or six hours after

Fig. 1.



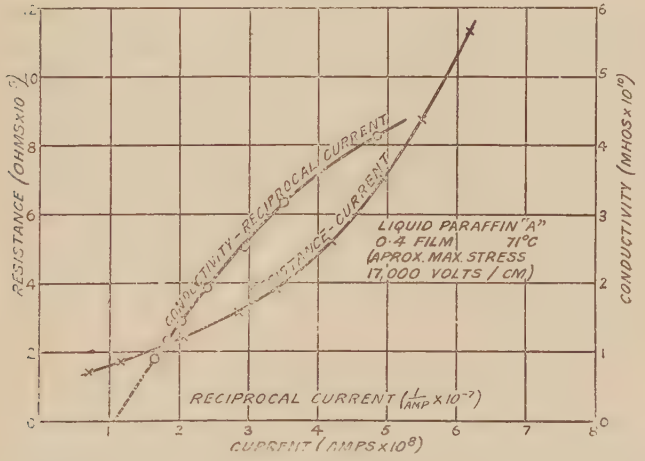
changing the applied potential, and this fact must be borne in mind if misleading results are to be avoided.

Our further experiments show that in general the resistance-current relationship is linear, thus supporting the evidence of the previous paper. This cannot, however, be expected to be always the case, for both the gas theory

* In expressing results of this kind I is usually graphed against V, but this is not such a satisfactory method as graphing R against I, for in the former case the curve is parabolic, and this can give, and has in some instances apparently given, an impression that a saturation current is being approached, when such is far from being the case.

and a contact-layer theory conclude that a straight line relationship holds only so long as the region of higher potential gradient at the electrodes extends over a distance which is small in comparison with the distance between them. We have been able to observe such departures from the linear law by applying sufficiently high stresses to *sufficiently thin films*. Fig. 2 shows some results obtained on a film of 0·4 mm. thickness, the maximum stress being 17,000 volts/cm. The possible existence of saturation currents is best seen by graphing the conductivity against the reciprocal of the current, as has been

Fig. 2.



done in the same figure. Assuming that extrapolation is justifiable, the form of the curve shows clearly that the maximum possible value of the current in this case is approximately 10^{-7} amp.

According to both the gas theory and a contact-layer theory, in all cases where the resistance-current relationship is linear the liquid between the electrodes remains unchanged except in the comparatively thin regions adjacent to the electrodes ; and in the gas theory it is the changes in the concentration of the "ions" at the electrodes which alters the total resistance between them. Some writers [6], [7], while basing their statements on the gas theory, make the assumption that the potential gradient is approximately constant throughout the

volume between the electrodes, and they apparently consider that the decrease in current on applying the potential is caused by a diminution in the number of ions everywhere within the film. It is clear that, given the withdrawal of a certain number of ions, the increase in resistance is much greater if they are all drawn from regions close to the electrodes than if they are withdrawn uniformly throughout the whole volume.

By making the assumption that the gradient is constant Nikuradse [7] obtains a formula which may be written in the form

$$I = a - b/R^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As can be seen by making R infinite, a is the value of the saturation current, and thus, if two values of R and I are known, the saturation current can be calculated. If the values of I in fig. 1 when R is equal to 4×10^{10} and 8×10^{10} ohms are inserted in the formula, the value of the saturation current obtained is 5×10^{-10} amp. As seen from the figure a value of $I = 8{\cdot}5 \times 10^{-10}$ amp. was measured in this instance, and the fact that the linear relationship still holds at this point shows that we were not even approaching a saturation value. Therefore we may state that our experimental results give no support to such an assumption *.

Absence of Residual Discharge Currents.

In the previous paper it was pointed out that charges would be expected to accumulate at the boundaries of any resistance layers, although, on account of the apparent absence of any residual discharge currents, it was considered that any effects due to such charges must be small in comparison with the actual changes in resistance. Since all resisting layer theories, including the gas theory, require the existence of space-charges, it was important to see whether this apparent absence of discharge currents was due to the space-charges disappearing in so short a period of time that they had previously escaped detection.

Measurements were therefore taken with the string galvanometer in which a potential was applied across a

* The values of mobility etc. mentioned in the article concerned are not invalidated by the assumption, since the necessary value of the saturation current is obtained by extrapolation from the experimental results, and not by means of the equation.

film until steady conditions had been reached, when the potential was then removed and the film short-circuited through the galvanometer. The usual practice of allowing the capacity discharge current to make a deflexion to indicate the moment of switching was not followed, as such a deflexion might have confused any other discharge effects. Accordingly the galvanometer was completely short-circuited during the capacity-discharge interval, but it was known from other experiments that this period was not greater than 0·05 sec. Curve 1, fig. 3 (Pl. VI.), shows the results of such an experiment. The initial value of the current in this case was 19×10^{-8} amp., and the steady value, which gave the deflexion shown on the left, was $5{\cdot}1 \times 10^{-9}$ amp. It is seen that any discharge effects existing after 0·05 sec. are extremely small, even in comparison with the final conduction current, which is itself less than one-thirtieth of the initial current. Therefore we conclude that any residual discharge effects which may exist disappear in a time of the same order as that of the capacity discharge, and that the law of superposition does not hold for liquid dielectrics.

Residual Effects observed after Removal of Potential.

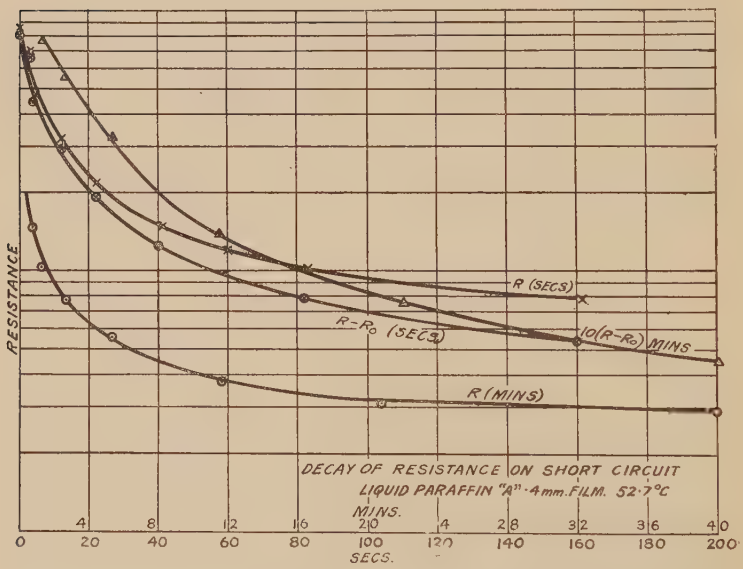
It is generally considered that, if after removing the applied potential a film of oil is short-circuited, it will regain its original conductivity *. In the previous paper these facts were accounted for by supposing that the contact resistances at the electrodes gradually disappeared, and a point in the theory was the assumption that the rate at which a contact resistance at an electrode disappears is proportional to that resistance. Thus, if resistances are formed at both electrodes, the curve connecting the total increase of resistance above the initial value with the time of short-circuit would be expected to be the sum of two exponentials.

Attempts were made to measure the rate at which the resistance disappeared on short-circuiting the film for definite intervals of time after steady conditions had been attained, and then reapplying the potential in the usual way. Intervals of short-circuit up to about 10 secs. were measured on the string galvanometer record and

* But see Bruninghaus[3].

longer intervals by means of a stop-watch. A typical result is shown in fig. 4, where both the total resistance, R, and the increase in resistance above the initial value, $R - R_0 = r$, are graphed against the time of short-circuit; but we have not been able to find a simple formula to express the relationship between them. In our experiments the period of short-circuit necessary for r to fall to $1/\varepsilon$ of its value, r, at the beginning of the period was found to vary from 10 secs. to 7 mins., its value depending on the nature and condition of the oil.

Fig. 4.

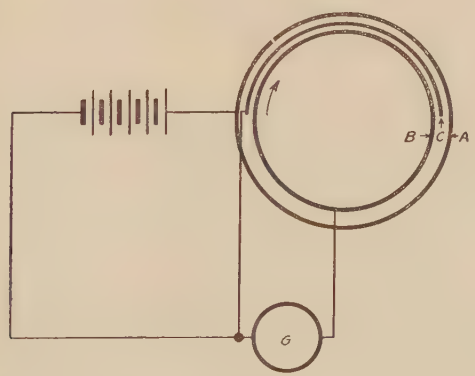

RESISTANCE
R (SECS.)
R-R₀ (SECS.)
10(R-R₀) MINS.
R (MINS.)
DECAY OF RESISTANCE ON SHORT CIRCUIT
LIQUID PARAFFIN "A" · 4 mm. FILM. 52·7°C
MINS.
4 8 12 16 20 4 28 32 36 40
0 20 40 60 80 100 120 140 160 180 200
SECS.

That the recovery of the initial onditions in a short-circuited film is not as simple as might at first be imagined is shown by the shape of the current-time records obtained on reapplying the potential. In fig. 3 (Pl. VI.), curve 2, is shown a record obtained on reapplying the potential across a film of liquid paraffin " B " after a short-circuit of 20·4 secs. These irregularities are not noticeable after very short intervals of short-circuit, but appear as the period is lengthened, and on further increasing the time the irregularities flatten out once more into a uniform curve,

gradually approaching the form of the curve observed when a potential is first applied (see curve 3, fig. 3 (Pl. VI.)). We have so far been unable to obtain a satisfactory explanation of these irregularities, but their existence is so marked that they must be considered when attempting to form any complete theory of the subject.

In an attempt to prove the existence or otherwise of contact resistances a special piece of apparatus was constructed. It consisted of two accurately concentric cylinders, A and B, the inner one, B, being capable of rotation through 180° and being shielded over one-half of its surface by a guard C as shown in fig. 5. This guard was placed close to B, but was insulated

Fig. 5.



Apparatus for detecting contact resistances and space-charges.

from it. Owing to constructional difficulties the distance between the cylinders could not be made particularly small, the actual value being approximately an eighth of an inch. The cylinders were placed in a jar of oil and connected to a battery and a galvanometer as shown. It was hoped that if, after a steady current value had been reached, B was rotated through 180°, any contact layer which had been formed upon its exposed half would be carried around behind the guard and a clean surface exposed, thus causing a decrease in resistance. On rotating B increases in current were observed, but such increases were masked in their initial stages by a discharge through the galvanometer in the opposite direction to the conduction current. Some typical results are given in

the following Table, where the change in resistance on rotation, with B negative and positive in turn, is expressed as a fraction of the total increase in resistance obtained from the final and initial current values.

Detection of Contact Resistances. Light Mineral Oil.

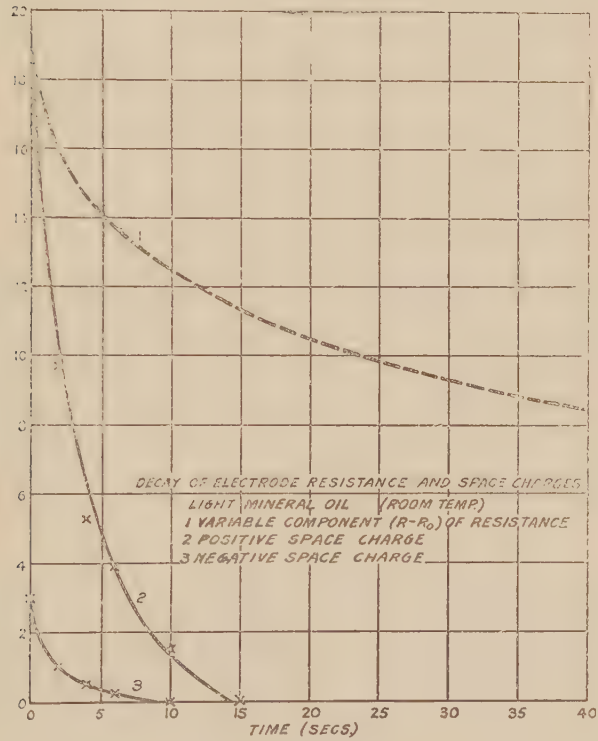
Temp.	Voltage.	$\frac{r}{R_0}$.	$\frac{r+}{r}$.	$\frac{r-}{r}$.
25° C.	350	0·81	0·29	0·14
34	350	0·44	0·34	0·22
34	240	0·70	0·43	—
38·7	240	0·34	0·28	0·09
48·9	240	0·24	0·32	0·22

It will be observed that the sum of the last two columns is always less than unity, which means that the sum of the resistances swept out was not great enought to account for the total resistance built up by the current.

The fact that a discharge took place through the galvanometer on rotating the inner cylinder of the above apparatus, shows that charges do accumulate in a liquid dielectric when a potential is applied across it. The presence of such charges seems to be required by the theories put forward, but, as we have seen, no discharge currents are observed under normal circumstances. Although no residual discharge currents are observed, we have shown that an oil film regains its conductivity at a measurable rate when it is short-circuited, and it is therefore, important to know whether the charge which we have shown to accumulate also decreases at a measurable rate, or whether it disappears with great rapidity. By rotating the inner cylinder after definite periods of short-circuit it was discovered that accumulated charges were still present after measurable intervals of time. In fig. 6 the magnitudes of the charges measured by means of a ballistic galvanometer are shown as a function of the time of short-circuit when the inner electrode was positive and negative respectively. The rate of decrease of r is shown on the same diagram for purposes of comparison, and it will be noticed that the decrease in r takes place at a slower

rate than the decrease in the charges. These experiments are of the greatest importance, as they show that resistance changes can exist in an oil film after the disappearance of the space-charges, and consequently any treatment which considers that the phenomena encountered in liquid dielectrics can be explained by changes in space-charges alone may lead to quite erroneous results.

Fig. 6.



We are also faced with the problem of accounting for the fact that space-charges can disappear within the body of a liquid between two electrodes when they are short-circuited without causing any effect in the external circuit. If we assume, in contrast to the assumption of material-resisting layers on the electrodes, that conduction in dielectrics is similar to that in gases, then the mobility of the "ions" is the same everywhere within the liquid.

as is also the dielectric constant. It is, then, not unreasonable to expect that on short-circuit the portion of an accumulated space-charge which will be attracted to an electrode would be just sufficient to neutralize the induced charge on it, and, such being the case, no residual discharge currents would flow in the external circuit. If there are two space charges of opposite sign they will partially neutralize each other on short-circuit, but the above argument is practically unaffected. If such a hypothesis is correct the objections previously raised to the gas theory are removed, and, in fact, it appears that, where no residual discharge currents occur, the increases in resistance due to the mechanism supposed for conduction in gases are more likely to be the kind involved than those due to layers of resisting material on the electrodes.

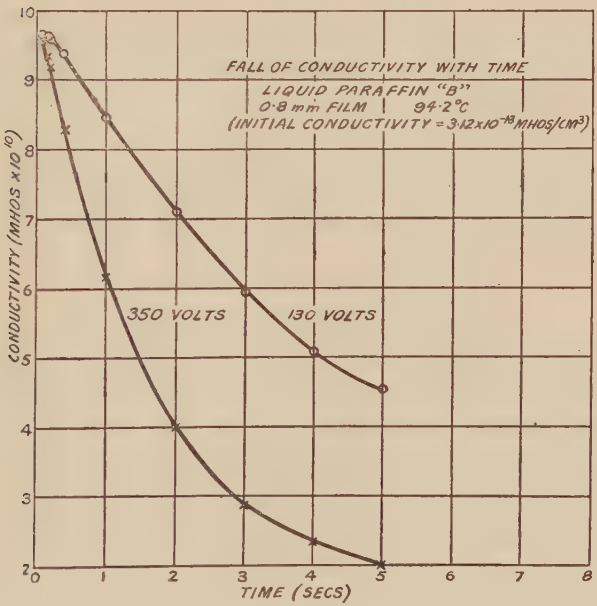
We have already pointed out that, according to the gas theory, regions of increased resistance occur near the electrodes owing to deficits in the total number of "ions" present, while space-charges occur through unequal deficits in the number of each sign. Upon short-circuit the space-charges will tend to disappear under the electrical forces involved, but the deficit in the total number can only be made up by the production of fresh "ions," and hence the decay of the additional resistance may take place at a much slower rate than the disappearance of the space-charges, as we have shown experimentally to be the case.

Initial Current Values.

We have taken numerous records with the string galvanometer of the decay of current with time from the moment of application of the potential across an oil film. A typical result is shown in fig. 3 (Pl. VI.), curve 3. In every case the shape of the curve indicates that the current is not the continuation of a capacity charging current such as would be obtained from a system of fixed capacities and resistances, but rather that it is due to the conductivity of the oil film decreasing with time from a definite initial value. Further, for a short space of time the curves are concave downwards, suggesting that at the moment of application of the potential the curves, apart from the true capacity current, would be practically horizontal. Fig. 7 shows the values of the conductivity calculated from two records with applied potentials of 350 and 130 volts, and from this

it is seen that the conductivity tends to the same initial value for both voltages. These results support the contention which has been put forward in the past *, that a film of liquid dielectric has a definite initial conductivity which is independent of the applied potential †. This initial value obviously corresponds to the true resistance, R_0, of the oil, as mentioned previously. The agreement between the initial values of the conductivity obtained from the string galvanometer records and the values of

Fig. 7.



R_0 obtained by extrapolation of the resistance-current curves for steady conditions is usually as good as can be expected, *e. g.*, from a record taken a few days after those giving the results in fig. 7, a value of $R_0 = 0{\cdot}93 \times 10^9$ ohms was obtained, while the value obtained by extrapolation of the resistance current curve was $1{\cdot}00 \times 10^9$

* *E. g.*, Tank [8].

† These results, together with those pointing to the absence of discharge currents, are quite different from those obtained by Whitehead [10] when dealing with solid and impregnated paper dielectrics.

ohms. In some cases, however, the agreement has not been satisfactory, and we hope to make further investigations on this point.

If, as our records indicate, the rate of decrease in conduction current is practically zero at the moment of application of the potential, the conductivity would not be expected to vary appreciably during a cycle of alternating current of a frequency of 50 or more. No variation of dielectric loss with frequency would therefore be expected, provided that the causes of such loss were the same as in direct current conduction. In general such losses are far from constant with changes in frequency, and therefore additional causes must be looked for. As shown by some workers [10] the presence of permanent dipoles in the liquid is a possible cause, but one would not expect its effects to be noticeable at comparatively low frequencies in non-viscous oils. The subject is of great theoretical importance, and we contemplate carrying out a series of measurements on the alternating current losses in oils over a range of frequencies from 500 cycles to 1,000 kilocycles.

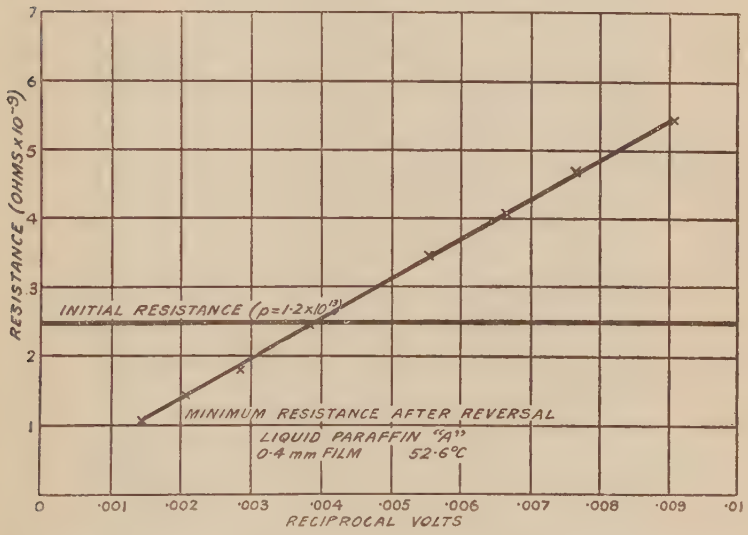
Reversal of Applied Potential.

It was previously thought that the value of the current immediately after the reversal of the applied potential was the same as that preceding the reversal. Results obtained with the string galvanometer show, however, that the current flowing after the reversal of the potential is always greater than that immediately before it, an example being shown in curve 1, fig. 8 (Pl. VII.). To check the obvious interpretations of these effects, as being due to a polarization potential having been set up or to some kind of rectifying action, a double reversal was made, the duration of the first reversal being less than one second. Curve 2, fig. 8 (Pl. VII..), shows a record of such an experiment, the duration of the first reversal being 0·33 sec., and it can be seen that the deflexion after the second reversal is greater than that due to the previous steady value of the current. Owing to switching complications it was found simpler to obtain the values of the current for the first reversal from a separate record. The values obtained from one experiment gave the following results :—

Steady current before reversal	$3{\cdot}50\times10^{-8}$ amp.
Extrapolated value of current at instant of second reversal	$5{\cdot}5\times10^{-8}$ amp.
Duration of first reversal	0·8 sec.
Current after 0·8 sec. on a single reversal	$5{\cdot}6\times10^{-8}$ amp.

If the sudden increase in current on reversal had been due to either of the causes mentioned above then the value of the current immediately after a second reversal would have been less than that observed for the first reversal. That such was not the case shows that some

Fig. 9.

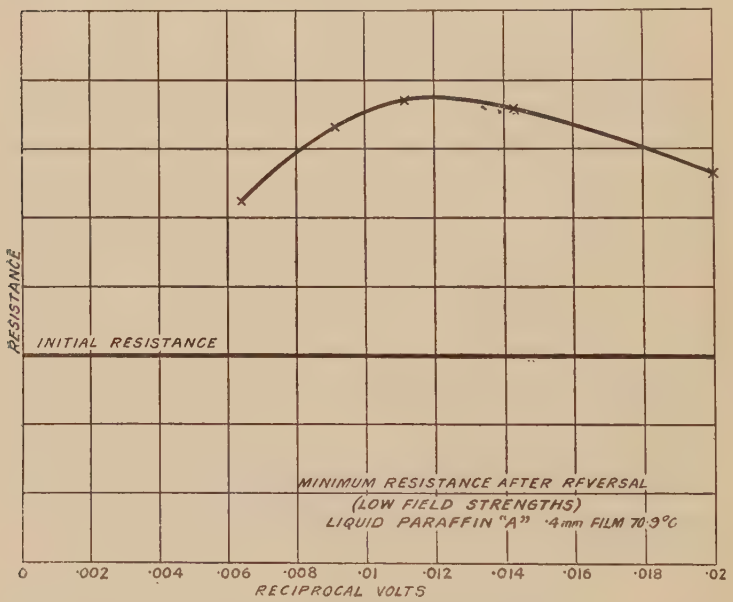


other explanation of the phenomena is required, and further investigation is called for.

In the previous paper were shown results obtained by the reversal of the applied potential, and it was seen that the current rose gradually to a maximum and then fell to a final steady value, two maxima being noted in some instances. The times to rise to the maximum values were usually of the order of minutes, but by suitably increasing the temperature and the applied stresses it was possible to record the changes by means of the string galvanometer without having to use too long a strip of bromide paper,

curve 3, fig. 8 (Pl. VII.), showing a typical result. From such curves it was noticed that the maximum value of the current was sometimes greater than the current on the initial application of the voltage, and from fig. 9, where the minimum values of the effective resistance are graphed against the reciprocals of the applied voltage, it can be seen that this maximum current may be more than twice the initial value. This shows that the increase in current is not simply due to a polarization potential assisting the

Fig. 10.



applied potential on reversal, for in that case the maximum current could not be greater than twice the initial current.

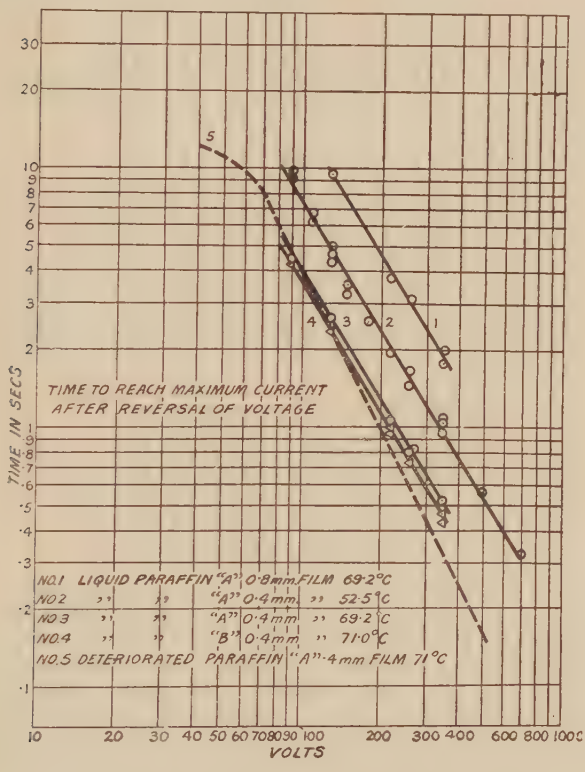
It is seen from fig. 9 that the relationship connecting the minimum values of the resistance, R_m, obtained from the reversal curves and the applied voltage, V, can be very well expressed by the formula

$$R_m = A + \frac{B}{V}, \quad \ldots\ldots \quad (4)$$

where A and B are constants. This linear relationship cannot, however, be expected to hold for low values

of V, for the proportionate changes in the current are small for such values, and therefore the value of R_m cannot continue to increase as $\frac{1}{V}$ is increased, but must gradually approach the initial value of the resistance, R_0. On account of the smallness of the currents and the length

Fig. 11.


TIME IN SECS
VOLTS
TIME TO REACH MAXIMUM CURRENT AFTER REVERSAL OF VOLTAGE
NO.1 LIQUID PARAFFIN "A" 0·8mm FILM 69·2°C
NO.2 " " "A" 0·4mm " 52·5°C
NO.3 " " "A" 0·4mm " 69·2°C
NO.4 " " "B" 0·4mm " 71·0°C
NO.5 DETERIORATED PARAFFIN "A" ·4mm FILM 71°C

of time involved it is difficult to obtain satisfactory records for the lower potentials. We have, however, obtained evidence that the linear relationship breaks down for low voltages, as is shown in fig. 10.

The earliest experiments showed that the time, t, taken for the current to reach its maximum value on reversal varied with the voltage, V. A formula has been found

expressing the relationship between them over the range of voltages from 130 to 700. This is

$$t = aV^{-n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where a is a constant depending on the condition of the oil, and n is a constant which was found to have the value $\frac{5}{3}$ for two different oils and for several ranges of temperature and film thickness, as seen in fig. 11. That the value of n is not always equal to $\frac{5}{3}$ was indicated by the results obtained when one of the two oils mentioned had become deteriorated. These are shown by the broken curve in fig. 11, where it can be seen that, over the voltage range mentioned, the curve is linear but that the value of n is greater than $\frac{5}{3}$.

The relationships expressed in formulæ (4) and (5) are of great interest, because they represent experimental laws which must be capable of explanation by any theory of the processes occurring in liquid dielectrics. It is easy to show that the more obvious methods of attempting to account for the phenomena are incapable of doing so. Up to the present we have been unable to find any satisfactory explanation.

Factors affecting Conductivity.

Examination of the mathematical treatment of the theory of conduction of electricity in gases leads to the following conclusions :—the relative absorption * increases as the rate of ionization decreases, it increases with the voltage, and it decreases as the mobility increases up to a certain critical value. Our experimental results lead to similar conclusions for conduction in oils, for it has been observed that (1) the relative absorption is greater the purer the oil (as judged by the initial conductivity), (2) the relative absorption becomes greater as the voltage is increased, and (3) absorption effects become less marked as the temperature is increased—that is, as the mobility increases—although no evidence has been obtained of any critical value. The agreement between these theoretical and experimental results affords considerable further support to the hypothesis that the process of conduction in liquids is of the same type as that in gases.

* Relative absorption is taken to be the ratio of the initial current to the final steady current.

So far we have not discussed the probable nature of the carriers by means of which an electrical current is conveyed through liquid dielectrics. It is well known that progressive purification of an insulating liquid effects large decreases in its conductivity, leading to the conclusion that the presence of impurities is in general the predominating factor in the conductivity of such liquids. The theory of conduction in gases assumes that ions are produced by the ionization of the molecules of the gas itself by some external source, and it is known that such ionizing agents as ν- and X-rays will increase the conductivity of liquid dielectrics. It does not seem reasonable, however, to assume that the normal conductivity of such liquids is due to an external source, such as some penetrating radiation, for in this case the majority of the ions would be produced from the molecules of the liquid itself, and the comparatively minute quantities of impurities present would not exert their predominating effect. Consequently any experiments carried out with external sources of ionization can only give information on the properties of the ions produced from the molecules of the liquid itself, and can have no bearing on the carriers normally involved. So far we have not obtained any direct evidence as to the nature of these carriers, but it seems clear that they must be produced mainly by internal causes.

Summary.

The possibility of applying the theoy of conduction in gases to conduction in liquid dielectrics is investigated. The gas theory is considered as leading to a distribution of internal resistances, and is shown to be a special case of a general theory advanced in a previous paper which attributed the fall of current usually known as " absorption " to the formation of internal resistances.

Experimental proof is obtained that in liquid dielectrics of the paraffin oil class space-charges are formed during the passage of the current and disappear slowly when the current is cut off. Nevertheless, discharge currents are found to be absent—an apparent contradiction which is resolved. Further residual effects are found which are quite independent of the space-charges, and are due only to the resistances set up by the current.

It is shown that the gas theory is able to account for all these facts, and is in some respects more capable of doing so than other theories. The law $R=R_0+nI$, arrived at by gas theory for final steady currents, is checked for a wide range of field strengths. Further agreements with gas theory are noted.

It is pointed out that some customary "approximate" treatments of gas theory lead to widely erroneous results, and cannot be used as tests of the gas theory. The cause of the failure of the "approximation" is elucidated.

Measurements have also been made at short time intervals after the initial application and after the reversal of the voltage. New phenomena have been discovered which do not appear susceptible of a simple explanation, although simple empirical laws have been found governing them.

Note.—Since this paper was written an article has appeared by Whitehead and Marvin [12] which deals with many of the points set out above. Many of their results are in good agreement with ours, but in some cases they differ somewhat sharply; and they do not attempt to give any explanation of some of the processes involved.

In conclusion, we wish to thank the International Telephone and Telegraph Laboratories Incorporated, for permission to publish the results set out in this paper.

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LXXVII. *An Acoustical Interpretation of the Schrödinger Wave Equation.* *By* R. B. Lindsay, *Associate Professor of Theoretical Physics, Brown University, Providence, Rhode Island, U.S.A.**

Abstract.

Investigation is made of the one-dimensional compressional vibrations of an ideal fluid of variable density (in equilibrium under the action of an external force) confined in an infinite tube of bounded but variable cross-section. The non-homogeneity, is so chosen that the wave velocity is everywhere the same. It is assumed that the ordinary acoustic equations are applicable except as modified by the non-homogeneity and that the cross-section is small enough so that transverse motions may be neglected. It is then found that the problem of determining the stationary wave-pattern in such a tube satisfying the boundary condition that the condensation $s=\delta\rho/\rho_0$ (where ρ_0 is the equilibrium density at any point and $\delta\rho$ the excess density) shall remain analytic, everywhere $<<1$ and vanish at infinity, is formally equivalent to the problem of solving a one-dimensional Schrödinger wave equation subject to the usual boundary condition. The appropriate density variation and the shape of the tube are related to the potential energy of the particle whose motion is described by the Schrödinger equation. As illustrations the free particle, the simple harmonic oscillator, and the one-dimensional hydrogen atom are investigated. The motion of the first corresponds to the oscillations of a homogeneous fluid in an infinite tube of uniform cross-section. For the simple harmonic oscillator the corresponding tube is a surface of revolution obtained by revolving the curve representing the error function about the x axis. A method of generalizing the results to the case of three degrees of freedom is indicated.

RECENT atomic physics is marked by the tendency to place less emphasis on a mechanical picture of atomic phenomena, and the use of models in describing atomic structure seems to be passing out of fashion, yielding place to more or less complicated mathematical algorithms based

* Communicated by the Author. This work was done while the writer was a member of the Department of Physics at Yale University.

on postulates which do not lend themselves easily to visualization in the mechanical sense. The logical validity of the methods employed in quantum mechanics is undoubted, and the success they have met with affords considerable justification. Nevertheless, many physicists continue to think in terms of pictures, generally mechanical, for this is the kind they have learned to understand best. The writer will not undertake to defend this attitude, for it is doubtless philosophically indefensible. Nevertheless, it is of interest to observe that a mechanical interpretation of Schrödinger's wave equation is perfectly possible, and, indeed, in many ways, though doubtless most of these will be unduly complicated to be of any utility. One such interpretation is due to Madelung*, who showed that the "time" wave equation can be put into the form of the hydrodynamical equation for the irrotational flow of a fluid. So far as the writer is aware this suggestion has never been followed up further. The recent work of A. Press (Phil. Mag. iv. p. 1248 (1927), and vi. p. 33 (1928)) should also be mentioned in this connexion. The present paper suggests a somewhat similar and rather simple interpretation based on the equations of acoustics.

We shall approach the matter by investigating approximately the vibratory motion of a compressible non-homogeneous fluid confined in an infinitely long rigid tube of bounded but varying cross-section. The non-homogeneity is assumed (for reasons which will appear presently) to be of such a character that

$$c = \sqrt{\mathrm{E}(x)/\rho_0(x)}$$

is a *constant*. The quantity c is, of course, the velocity of the compressional waves in the tube, and the volume elasticity $\mathrm{E}(x)$ and the equilibrium density $\rho_0(x)$ are now supposed functions of distance along the tube. We shall confine our attention at first to one-dimensional flow, choosing the diameter of the tube small enough so that transverse motions may be neglected (as in the horn theory of A. G. Webster†). We shall also assume that the ordinary equations of acoustics are applicable except as modified by the lack of homogeneity.

The equation of continuity for one-dimensional motion is

$$\mathrm{S}\dot{\rho} = -\frac{\partial}{\partial x}(\rho \mathrm{S}\dot{\xi}), \quad \ldots\ldots \quad (1)$$

* E. Madelung, *Zs. f. Phys.* xl. p. 322 (1926).
† A. G. Webster, Proc. Nat. Acad. Sci. v. p. 275 (1919).

where ρ is the density, S the cross-sectional area, and ξ the particle velocity. Introducing the condensation s, *i.e.*, placing $\rho=\rho_0(1+s)$, where $\rho_0(x)$ is the equilibrium density at any point, and assuming that everywhere $s<<1$ at all times, we may write in place of (1)

$$\rho_0 S\dot{s}=-\frac{\partial}{\partial x}(\rho_0 S\dot{\xi}), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

wherein $\frac{\partial}{\partial x}(\rho_0 s S\dot{\xi})$ has been neglected.

We must next write the equation of motion. In order to assure equilibrium for the undisturbed fluid in the tube it is essential that an external force act on the fluid in the x direction (consider as an illustration a fluid in equilibrium under gravity in which density and pressure increase downward). We may represent this force per unit volume as the gradient of the equilibrium pressure, viz., $\frac{\partial p_0}{\partial x}$. The equation of motion for small disturbances is then

$$\rho\ddot{\xi}=-\frac{\partial(p-p_0)}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where p is the absolute pressure. The difference $p-p_0$ is then the acoustic excess pressure δp, which we shall more conveniently denote by p_e. With the same convention as above regarding s, (3) may then be written

$$\rho_0 S\ddot{\xi}=-S\frac{\partial p_e}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we differentiate (2) with respect to the time and substitute from (4), the result is

$$\rho_0 S\ddot{s}=S\frac{\partial^2 p_e}{\partial x^2}+\frac{\partial S}{\partial x}\cdot\frac{\partial p_e}{\partial x}. \quad . \quad . \quad . \quad . \quad (5)$$

As usual, we assume a relation between excess pressure $p^e=\delta p$ and the corresponding density alteration $\delta\rho$ in the form

$$\delta p=c^2\delta\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where, in accordance with our previous assumption, c is to remain constant. Since by definition $s=\delta\rho/\rho_0$, it follows that

$$s=p_e/\rho_0 c^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and hence equation (5) becomes

$$\frac{\partial^2 p_e}{\partial x^2}+\frac{1}{S}\cdot\frac{\partial S}{\partial x}\cdot\frac{\partial p_e}{\partial x}=\ddot{p}_e/c^2, \quad . \quad . \quad . \quad (8)$$

or, if p_e is a harmonic function of the time with frequency $\nu = \omega/2\pi$, the excess pressure equation takes the form (reverting to ordinary derivatives)

$$\frac{d^2p_e}{dx^2} + \frac{1}{S}\cdot\frac{dS}{dx}\cdot\frac{dp_e}{dx} + \frac{\omega^2}{c^2}p_e = 0. \quad . \quad . \quad . \quad (9)$$

For any particular form of tube and any given type of variation in ρ_0 we are restricted to solutions of (9), for which s is very small compared with unity no matter how long the tube may be.

Let us now consider the Schrödinger wave equation for one degree of freedom, viz.,

$$\frac{\partial^2\psi}{\partial x^2} + (\alpha - \mu V)\psi = 0, \quad . \quad . \quad . \quad . \quad (10)$$

where $\alpha = 8\pi^2 m_0/h^2 . E$ and $\mu = 8\pi^2 m_0/h^2$, E being the total energy of the particle whose motion is being considered, and V the potential energy. To establish an acoustical analogy whereby equation (10) may be looked upon as an acoustic wave equation, let us consider ψ as made up of two parts, *i. e.*,

$$\psi = \sqrt{S}.p_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where S is the area of cross-section of a tube such as has been earlier described and p_e is the excess pressure due to acoustic propagation through this fluid. On substitution into (10) and reversion to ordinary derivatives, we have

$$\frac{d^2p_e}{dx^2} + \frac{1}{S}\cdot\frac{dS}{dx}\cdot\frac{dp_e}{dx} + \frac{1}{\sqrt{S}}\frac{d^2\sqrt{S}}{dx^2}\cdot p_e + (\alpha - \mu V)p_e = 0. \quad (12)$$

We shall now suppose that $\sqrt{S}$ is given as the first eigenfunction of the equation

$$\frac{d^2\sqrt{S}}{dx^2} + [f(\mu) - \mu V]\sqrt{S} = 0, \quad . \quad . \quad . \quad (13)$$

where $f(\mu)$ is the first eigenwert *. This step is dictated by the fact that the acoustical analogy demands that S shall be finite, continuous, and single-valued everywhere, and vanish at infinity. Furthermore, the first eigenwert is chosen in order to secure a tube which has no knots between $x = 0$ and

* We are applying the analysis to problems in which *discrete* eigenwerte exist, though it will probably prove unnecessary to make this restriction general.

$x=\infty$. With this assumption, the original equation now becomes

$$\frac{d^2p_e}{dx^2}+\frac{1}{S}\cdot\frac{dS}{dx}\cdot\frac{dp_e}{dx}+[\alpha-f(\mu)]p_e=0, \quad . \quad . \quad (14)$$

which is identical with equation (9) if $\alpha-f(\mu)=\omega^2/c^2$. The problem of solving the one-dimensional Schrödinger equation is thus shown to be equivalent to the study of the small acoustic harmonic oscillations in a non-homogeneous fluid confined in an infinitely long tube with variable cross-sectional area given by equation (13). In particular we are restricted to those oscillations for which $s \ll 1$, and for which both s and $\dot{\xi}$ vanish at ∞. The choice of variation in density ρ_0 is naturally somewhat arbitrary, but in the cases so far studied (see below) it has been found most simple analytically to choose ρ_0 so that $\rho_0\sqrt{S}=$ constant. In this case we are at once restricted to solutions of (14) which correspond to the eigenwerte of the corresponding Schrödinger equation, for it makes ψ proportional to the condensation s. This is an appropriate place to point out that, while the introduction of two varying factors (*i. e.*, both S and ρ_0) may seem to involve an unduly artificial construction, simple calculations indicate that no simple interpretation of the Schrödinger equation is possible from the acoustic point of view if (1) ρ_0 is kept constant and S varies, or (2) if S is kept constant and ρ_0 varies. The point of view chosen in the present paper is thus not quite so arbitrary as might at first be thought.

Proceeding to special problems, consider first the trivial case of a free particle. Here $V=0$, and the equation (13) becomes, since the first eigenwert may here be considered zero (they are not discrete),

$$\frac{d^2\sqrt{S}}{dx^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

leading at once to $S=$ constant. Thus, free motion corresponds to the oscillations of a homogeneous fluid in an infinite tube of *uniform* cross-section.

Next, consider the more important problem of the simple harmonic oscillator. Here $V=Kx^2$, and the equation (13) becomes

$$\frac{d^2\sqrt{S}}{dx^2}+[f(\mu)-\mu Kx^2]\sqrt{S}=0, \quad . \quad . \quad . \quad (15)$$

of which the first eigenwert* is $f(\mu)=\sqrt{\mu K}$, and the

* A. Sommerfeld, *Wellenmechanischer Ergänzungsband*, pp. 18, 19.

corresponding eigenfunction is $\sqrt{S} = Ce^{-\sqrt{\mu K}x^2/2}$, so that for the shape of the tube we have

$$S = C^2 e^{-\sqrt{\mu K}x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The equation (14) becomes

$$\frac{d^2 p_e}{dx^2} - 2\sqrt{\mu K}x.\frac{dp_e}{dx} + [\alpha - \sqrt{\mu K}]p_e = 0. \quad . \quad (17)$$

Assuming now $\rho_0\sqrt{S} = \text{const.}$, we have

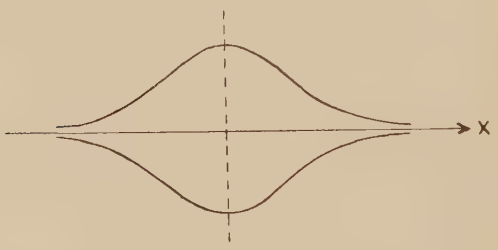
$$\rho_0 = \rho_{01} e^{\sqrt{\mu K}x^2/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

From equation (7) it is evident that we must restrict ourselves to polynomial solutions of (17). That is, we have

$$(p_e)_n = A_n H_n(\sqrt[4]{\mu K}x), \quad . \quad . \quad . \quad . \quad (19)$$

where H_n is the Hermite polynomial of order n and A_n

Fig. 1.



a normalized constant. For the condensation we have consequently

$$s_n = \frac{A}{\rho_{01} c^2} H_n(\sqrt[4]{\mu K}x) e^{-\sqrt{\mu K}x^2/2}, \quad . \quad . \quad . \quad (20)$$

which, of course, has the same form as the Schrödinger ψ_n aside from a constant multiplier. From (16) the tube is seen to be a surface of revolution obtained by revolving the curve representing the error function about the axis of x (fig. 1). The solution given by (19) and (20) represents a standing wave-pattern in the infinite tube resulting from the imposed boundary conditions. There will be points of zero condensation (or loops) at the zeros of $H_n(\sqrt[4]{\mu K}x)$ and nodes (maximum condensation) at the maxima of the polynomials. The eigenwerte (see Sommerfeld, *loc. cit.*) for this problem are given by

$$\alpha / \sqrt{\mu K} = 2n + 1, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where n is an integer, and the corresponding characteristic wave-lengths of the vibrations in the tube are given by

$$\omega^2/c^2 = 4\pi^2/\lambda^2 = \alpha - \sqrt{\mu}\mathrm{K}. \quad . \quad . \quad . \quad . \quad (22)$$

For $n=0$ we have $\omega/c=0$, corresponding to infinite wave-length or zero frequency (for c finite); this corresponds to the ground state of the oscillator. For $n=1$, we have

$$\lambda^2 = \frac{\pi}{\sqrt{\mu\mathrm{K}}} = \frac{h}{4\pi m_0 \nu_0}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

since (Sommerfeld, *loc. cit.*) $\sqrt{\mu\mathrm{K}} = 4\pi^2 m_0 \nu_0/h$, where m_0 is the mass of the oscillator and ν_0 is the fundamental mechanical frequency. For an intra-atomic frequency (m_0 being the mass of the electron) $\nu_0 = 10^{16}$, we get λ of the order of 10^{-8} cm. In this case the medium must be extremely fine-grained. Knowledge of the corresponding frequency will depend on the value assigned to c. If the latter is given the value of the velocity of light, the above, of course, corresponds roughly to hard X-rays or gamma radiation.

As another illustration, let us consider the one-dimensional hydrogen atom, for which $V = -Ze^2/x$, Z being the number of positive charges on the nucleus. The equation (13) now becomes

$$\frac{d^2\sqrt{\mathrm{S}}}{dx^2} + [f(\mu) + \mu Ze^2/x]\sqrt{\mathrm{S}} = 0. \quad . \quad . \quad . \quad (24)$$

The first eigenwert is $f(\mu) = -(\mu Ze^2/2)^2$, and the corresponding eigen-function is

$$\sqrt{\mathrm{S}} = \mathrm{C}x e^{-\frac{\mu Ze^2 x}{2}},$$

so that for the cross-sectional area of the tube we have

$$\mathrm{S} = \mathrm{C}^2 x^2 e^{-\mu Ze^2 x}, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where negative values of x have here no significance. The tube is a surface of revolution obtained by revolving the curve given by $\sqrt{\mathrm{S}}$ about the x axis, the appearance of the cross-section being shown in fig. 2. The maximum cross-section occurs for $x = 2/\mu Ze^2$. The equation (14) becomes for this case

$$\frac{d^2 p_e}{dx^2} + (2/x - \mu Ze^2)\frac{dp_e}{dx} + [\alpha + (\mu Ze^2/2)^2]p_e = 0. \quad (26)$$

Assuming now, as before, $\rho_0\sqrt{\mathrm{S}} = \text{const.}$,

$$\rho_0 = \frac{\rho_{01}}{x} \cdot e^{\frac{\mu Ze^2 x}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

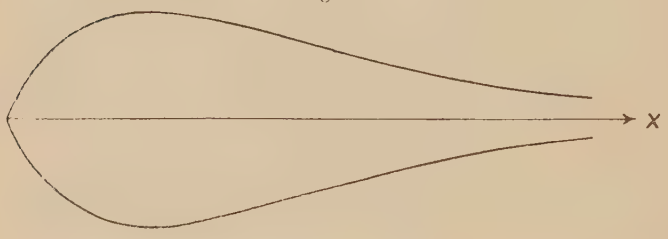
The solutions of (26), which correspond to the eigenwerte of the Schrödinger equation and which assure that s and $\dot{\xi}$ shall vanish at infinity, are found to be

$$(p_e)_n = \frac{A_n}{x} \cdot e^{\frac{n-1}{2n}\mu Ze^2 x} B_n(x), \quad . \quad . \quad . \quad (28)$$

where the $B_n(x)$ form a set of polynomials, the solutions of the equation

$$\frac{d^2v}{dx^2} - 2\sqrt{-\alpha} \cdot \frac{dv}{dx} + \frac{\mu Ze^2}{x} v = 0, \quad . \quad . \quad . \quad (29)$$

Fig. 2.



corresponding to the values of α which are eigenwerte of the original Schrödinger equation. Thus we have

$$\left.\begin{aligned} B_1(x) &= x, \\ B_2(x) &= x\left[1 - \frac{\mu Ze^2}{4}x\right], \\ B_3(x) &= x\left[1 - \frac{\mu Ze^2}{3}x + \frac{1}{6}\left(\frac{\mu Ze^2}{3}\right)^2 x^2\right], \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \quad . \quad . \quad (30)$$

The characteristic wave-lengths for this case are given by

$$x = \frac{2\pi}{\sqrt{\alpha + \left(\frac{\mu Ze^2}{2}\right)^2}}. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

For the electron $\mu Ze^2/2$ is of the order of $10^8 Z$, and hence, since the eigenwerte are given by

$$\alpha = -\left(\frac{\mu Ze^2}{2n}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

the first finite wave-length is again of the order of 10^{-8} cm.

We may now suggest the following generalization to three degrees of freedom. For this case the equation of continuity may be written

$$\dot{\rho}+\frac{\partial(\rho_0\dot{\xi})}{\partial x}+\frac{\partial(\rho_0\dot{\eta})}{\partial y}+\frac{\partial(\rho_0\dot{\zeta})}{\partial z}=0, \quad . \quad . \quad (33)$$

where the component displacements along the three coordinate axes are ξ, η, and ζ respectively, and we are, as usual, assuming that $s<<1$. Now the component equations of motion are to the same approximation,

$$\left.\begin{aligned}\rho_0\ddot{\xi}&=-\frac{\partial p_e}{\partial x},\\ \rho_0\ddot{\eta}&=-\frac{\partial p_e}{\partial y},\\ \rho_0\ddot{\zeta}&=-\frac{\partial p_e}{\partial z},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

if we neglect terms of the order of $\dot{\xi}\frac{\partial\dot{\xi}}{\partial x}$, etc. The continuity equation then becomes

$$\rho_0\ddot{s}=\nabla^2 p_e,$$

or, since $p_e=\rho_0 c^2 s$ and c is still assumed to be constant we have

$$\nabla^2 p_e+\frac{\omega^2}{c^2}p_e=0 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

for the excess pressure equation (p_e being harmonic in time with frequency $\omega/2\pi$). To this must be added a statement of the condition that there must be zero velocity normal to the confining surface at every point. If we denote the direction cosines of the normal at any point by n_x, n_y, n_z respectively, the condition becomes

$$n_x\dot{\xi}+n_y\dot{\eta}+n_z\dot{\zeta}=0 \quad . \quad . \quad . \quad . \quad . \quad (36)$$

at the bounding surface.

From the equations of motion (34) it follows that (36) can be written

$$n_x\frac{\partial p_e}{\partial x}+n_y\frac{\partial p_e}{\partial y}+n_z\frac{\partial p_e}{\partial z}=0. \quad . \quad . \quad . \quad (37)$$

Now consider the Schrödinger equation for three degrees of freedom, namely,

$$\nabla^2\psi + (\alpha - \mu V(x,y,z))\psi = 0. \quad . \quad . \quad . \quad (38)$$

As before, let us introduce the substitution of the form

$$\psi = F(x,y,z) \cdot p_e, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where the significance of F will appear in a moment. The equation (38) now becomes

$$\nabla^2 p_e + (\alpha - \mu V)p_e + \frac{\nabla^2 F}{F} \cdot p_e + \frac{2}{F} \cdot \sum_{x,y,z} \frac{\partial F}{\partial x} \cdot \frac{\partial p_e}{\partial x} = 0. \quad (40)$$

Now let $F(x,y,z) = \text{const.}$ be the equation of the surface within which the fluid is confined. Then, since $n_x \propto \frac{\partial F}{\partial x}$, etc.,

$$\sum_{x,y,z} \frac{\partial F}{\partial x} \cdot \frac{\partial p_e}{\partial x} = 0$$

is precisely the condition that there shall be no motion normal to the confining surface. We shall suppose that F is determined as the first eigenfunction of the equation

$$\nabla^2 F + [f(\mu) - \mu V]F = 0, \quad . \quad . \quad . \quad . \quad (41)$$

where $f(\mu)$ is the first eigenwert of the problem. The equation (40) then becomes

$$\nabla^2 p_e + [\alpha - f(\mu)]p_e = 0, \quad . \quad . \quad . \quad . \quad (42)$$

which, if $\alpha - f(\mu) = \omega^2/c^2$, is precisely the equation (35) for the excess pressure in the fluid vibrations within the surface. The solutions of (42) are to be taken as those which correspond to the eigenwerte of the original Schrödinger equation (38). In the acoustical problem these assure the vanishing of s and the component velocities at infinity, as well as their analytic nature everywhere else.

LXXVIII. *Interference Methods and Stellar Parallax.* *By* E. H. Synge *.

THERE is no difficulty in seeing that the methods of stellar interferometry should afford a means of determining parallax with a refinement comparable with the resolving power of the interferometer. If the images of the field provided by the two sets of mirrors of a stellar interferometer, whether in the original form of Michelson or in that proposed by the writer in a recent paper †, are exactly superposed in the focal plane of the telescope, and if, for any one star S, the path-lengths of the two beams are made equal, there will generally be inequality of path-lengths for any other star S′. If ϕ (supposed small) is the component of the angular distance between S and S′ along the axis of the interferometer, then, if the path-lengths are equal for S, the relative retardation between the beams in the case of S′ will be proportional to ϕ, and if this retardation is known ϕ will be known. This obviously involves a method of determining parallax. For if we have some convenient means of finding this retardation we shall know ϕ, and from a number of observations, at different times and with different pairs of stars, the parallaxes can be calculated. This idea, in a somewhat modified form, will be applied to the design of an apparatus in the present paper.

First, however, a method will be suggested for obtaining interference between two beams from a star which differs from Michelson's, and which seems to possess great advantages over it. The interference fringes used by Michelson are diffraction phenomena, but the principle ‡, which shows how different fringes are formed by different points of light, is one of geometrical optics. And as this is the vital factor, it is clear that the method does not essentially depend upon diffraction at all. Interference of the two beams can be obtained independently of diffraction by means of a half-silvered plate, the two

* Communicated by the Author.

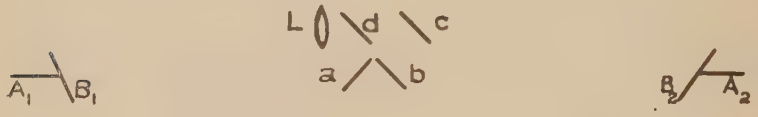
† "A Modification of Michelson's Beam Interferometer." Phil. Mag. (August 1930).

‡ Schuster and Nicholson, 'Theory of Optics,' 3rd ed. pp. 167 F, G.

beams being completely superposed on one another, as in Michelson's ordinary interferometer for measuring wave-lengths ; and in that form the interference is adapted to measurement by a photo-electric cell.

It was for this reason that the writer was led to consider this form of interference as an alternative to the diffraction fringes used by Michelson. But other advantages soon became apparent which suggested that the method might well supersede Michelson's for every purpose. The diagram shows an arrangement by which this direct interference could be effected.

A_1, A_2 are the axes of two siderostats, these axes being collinear. B_1, B_2, the mirrors of the siderostats, are set so that the two beams from a certain star are reflected in opposite directions along the common polar axis. a, b, c are fully silvered mirrors, and d is a half-silvered mirror. The beam from B_1 is first reflected by a, and then, in part, by d, into a telescope L. The beam from B_2 is reflected by b to c, and then passes, in part, through d, entering L in



the same direction as the beam from B_1, upon which it is exactly superposed. Since the beam from B_2 passes through the thickness of the glass of d, a plate of glass e (not shown) of similar thickness should be interposed at the same angle in the path of the beam from B_1.

Disregarding, for the present, the dispersion of the atmosphere, as is usually but unjustifiably done, the phenomena will be the following.

If the path lengths of these two beams, as measured from any arbitrary wave-front in space, are equal as they enter L, the two beams will reinforce one another, and all the light energy of both will pass into L. But if the beams differ by a half wave-length they will cancel one another, and an observer looking through the telescope will not perceive anything. Under good atmospheric conditions an observer will perceive an image of the star constantly vanishing and re-appearing, as the optical path-lengths of the two beams keep varying slightly. If, however, one

path-length becomes longer than the other by more than a few wave-lengths the different colours, for ordinary mixed light, will produce an overlapping, and the fluctuations will diminish and finally cease altogether, the star then appearing of an unvarying brightness, which is half the brightness when the two path-lengths are exactly equal.

It has been assumed that the star in question is sufficiently beyond the limit of resolution of the interferometer. For as the star approaches resolution, the maxima and minima will become less and less marked, even in the immediate neighbourhood of path-length equality. The optical principle here is essentially the same as in the case of the fringes used by Michelson. For if we consider two very close stars, the difference between whose declinations is equal to $\frac{\lambda}{2D\cos\theta}$, where D is the distance between the central points of the two siderostats, θ the declination of one of the stars, and λ a wave-length, the same general law which determines the phenomena exhibited by Michelson's fringes in the case of a double star * shows that if there is exact equality of path-lengths in the case of one of the stars, and thus mutual reinforcement of the two beams from it as they enter L, there will be a difference of $\frac{\lambda}{2}$ between the path-lengths of the beams from the other star, and therefore no light from it will enter L. It is merely a matter of integration to extend the case of a double star to that of a stellar diameter, and to find the decreasing sharpness of fluctuation as resolution is approached. When resolution is complete no fluctuation at all will be observed. This corresponds to the vanishing of Michelson's fringes.

In the case of a star bright enough to give a graphical record with a photo-electric cell, the recording apparatus should show a series of maxima and minima readings, of which several hundreds would be registered in a few minutes. The highest maxima would correspond to path-equality, and their difference from the mean would indicate with quantitative exactness the approach to resolution.

The method suggested for the equalization of path-

* Schuster and Nicholson, *loc. cit.*

lengths, by means of the mirrors in the diagram, is the following. The four mirrors *a*, *b*, *c*, *d*, with the telescope L, are mounted on a carriage, which can travel on rails laid down between the two siderostats, the rails following a polar axis. A first approximation to equality of path-lengths is obtained by moving the carriage. To effect and maintain exact equality, *a* and *d* are rigidly connected and movable together relative to the carriage in the direction of a polar axis, while *b* and *c* are rigidly connected and capable of receiving slight rotations together, about an axis perpendicular to the plane of the paper, which represents a vertical section. Equality of path-lengths is obtained by moving *a*, *d* to the proper position, and maintained by very small rotations given to *b*, *c*, such rotations changing slightly the path-length of the beam from B_2, while not altering its direction, as it leaves *c*. This, for instance, will always remain parallel to its direction as it leaves B_2, if *b* and *c* are parallel to one another.

These methods of equalizing the path-lengths do not, however, by any means cover the question. Owing to the different refraction by the atmosphere of light of different wave-lengths the rays of different wave-lengths from a star reach the surface of the earth at slightly different angles, and in all forms of the interferometer this translates itself into a difference in relative retardation when the two beams unite. Thus, if there is equality of path-length at L for light of one wave-length, there will in general not be equality for light of a different wave-length. In amount the effect depends upon the altitude of the star, being zero at the zenith. It also increases with the length of the base-line, and the consideration of it is vital in any form of the stellar interferometer, although it is not referred to at all in most accounts of Michelson's Beam Interferometer. In that apparatus the crossed glass wedges and tilted plates set in the paths of the two beams can supply compensation and produce tolerable achromatism for the short base-lines used; and the writer fancies that this is usually done in an empirical way by skilful manipulation. The compensation is, of course, produced by one beam having a greater air-path and a less glass-path than the other. Equality of path-length can thus in general be obtained for any two different wave-lengths, and this implies an approximate achromatism.

For the much longer path-lengths feasible by the use of two siderostats it is, however, necessary to go into the question more closely, and to provide means by which the proper compensation may be given for any case by a calculated adjustment. A consideration of the question will also lead us to the recognition of a system of interference bands or fringes, which have many advantages over Michelson's fringes, and which seem to afford the only means of extending the method to those very long base-lines whose employment the writer, in his previous paper, suggested was conceivable.

The simplest method to secure chromatic compensation is to rotate e from its standard position, through a calculated angle. If the dispersion of e (with d) is well chosen quite good compensation should be obtained in this way. But a better compensation could be obtained by the following arrangement of crossed wedges. We will suppose that a plate of glass v_1 and a pair of crossed wedges w_1 are set at right angles in the path of the beam from B_1, while a plate v_2 and crossed wedges w_2 are similarly set in the path of the beam from B_2. v_1 and w_2 are constructed of one variety of glass, and v_2 and w_1 of another variety having a different dispersion. When the thickness of w_2 is adjusted so as to be equal to that of v_1, and the thickness of w_1 to be equal to that of v_2, the systems affect both beams equally, and there is no compensating effect. This is the proper adjustment for a star in the zenith. We may now select some wave-length λ as a standard, and suppose that one wedge of the pair w_1 and one of w_2 are moved transversely in such a way that the path-lengths of the rays of wave-length λ vary equally, and thus remain equal. The angles of the wedges can be made so that this effect is produced by an equal transverse motion of the two wedges in question. w_1 and w_2 might be conveniently set in the paths of the respective beams, just before each of them reaches d, and it would then be easy to make the turning of a single screw T impart the proper relative motion to both wedges at the same time.

If now we consider a star which is not in the zenith, and suppose that the system is adjusted so that the rays of wave-length λ unite in L with equality of path-lengths, then, if the achromatizing system v_1, w_1, v_2, w_2 were absent, there would be a certain relative retardation r between

the rays of any other wave-length ν. But, owing to the different dispersions of w_1 and w_2, there will be one adjustment of w_1 and w_2, which, while preserving equality of path-lengths for λ, will produce a compensating retardation $(-r)$ for ν. The rays of both wave-lengths λ and ν will then unite with equality of path-lengths, and if the dispersions of the two glasses are appropriately chosen, there will be practical achromatism over a considerable part of the spectrum, provided the base-line is not too long and the star is not too far from the zenith. A more complicated achromatizing system, with several crossed wedges, would provide a more perfect achromatism, but we need not consider this here.

Even if perfect achromatism were secured for one instant it would not, however, persist without continual adjustment of the screw T, for the distance of the star from the zenith and the effect in the interferometer will be continually changing. The rate of this change will be a minimum for the proposed apparatus when the star is on the meridian, and stars can therefore be best examined when they are in the immediate neighbourhood of the meridian. During a few minutes while they are crossing the meridan no readjustment of T should usually be needed. The screw would be set, by calculation, for this passage, the calculation being based on the altitude of the star during the passage and on the actual refraction of the atmosphere, which, of course, will not always remain constant. The correctness of the adjustment could be tested with great simplicity by converting the image of the star into a short spectrum which will be traversed with bands if the adjustment is not correct. These are the bands or fringes which have been referred to as a substitute for Michelson's diffraction fringes.

In the method suggested above for the use of a photo-electric cell to determine stellar diameters the achromatism should be provided for the photo-electrically active portion of the spectrum, and only that part of the spectrum admitted to the cell for which the achromatism is good. This method of measuring stellar diameters should succeed best when the interferometer has about one-half the full resolving power of the star, and should give the diameter correctly to at least 1 part in 100 if a sufficiently broad beam is used and only the highest of the maxima

considered. When a photo-electric cell is employed, the complementary beam issuing from d in the direction of ad produced could be used by an observer for purposes of control.

We will now turn to the spectrum bands or fringes, the simplicity and importance of which may not be immediately apparent. The existence of these bands does not essentially depend upon refraction or defects in achromatism, for if we suppose that the system is perfectly achromatic, but that one beam has a retardation of r upon the other, then there will be a dark band across the spectrum at every wave-length which has the value $\frac{r}{n+\frac{1}{2}}$, where n is any integer.

To recognize the importance of these bands it is only necessary to observe that the same general law * which governs the phenomena of the Michelson fringes in the case of a double star operates equally here. If we consider the spectra given by a double star, the north-south component of the angular distance between whose members is equal to $\frac{\lambda}{2D\cos\theta}$, then, assuming that the spectra are ordinarily continuous, if one spectrum exhibits a dark interference band having its centre at the wave-length λ, the other spectrum will have λ as the centre of a bright band. And since these two spectra are superposed we shall have the same kind of phenomena as is shown by Michelson's fringes in a similar case. But everything will be exhibited more clearly here, because a double star will, in general, only be exactly resolved for a single wave-length, and for accurate results this wave-length must be known. In Michelson's fringes all the wave-lengths are confounded together. Here, on the other hand, the exact wave-length can be immediately picked out. The passage from a double star to stellar diameters is, as before, only a matter of integration, the resolution being complete when the bands vanish; and here again there is superiority to Michelson's fringes, for the completeness of the resolution should differ, according to a definite law, for different wave-lengths, and by any non-correspondence of the

* Schuster and Nicholson, *loc. cit.*

observed difference with theory, effects such as the reddening of the limbs of a star, due to the stellar atmosphere, should be immediately indicated.

In the case of a star whose spectrum is itself made up of noticeable lines an ordinary spectrum of the star (not exhibiting interference) should be obtained, and brought for comparison into the field of vision, side by side with the interference spectrum. Such a comparison spectrum would, indeed, be useful in any case. The question of the slits is not of great importance. No slit at all may be used, or a slit sufficient to delimit the star from the rest of the field, or a slit as in ordinary stellar spectroscopy; the choice depends upon the particular case and the amount of light available.

It is in regard to the amount of light available that we have one of the greatest advantages of the spectrum bands over Michelson's fringes; for in the case of the latter the light is wastefully spread over the area of a circle and enormous magnification has to be employed. Using the spectrum bands the light is, on the other hand, concentrated along a narrow line which need appear little broader than the limit of ocular resolution, and if the star is faint only two or three bands are necessary. The bands will of course move about very much like Michelson's fringes, but they will not become confused in the same way when one of the two beams is a few wave-lengths longer than the other. In such a case they will simply increase in number, and not very rapidly, so that the apparatus should be far easier to control by means of them. On the whole, after allowing for the loss of half the light at the half-silvered plate, there should be an advantage of about 100 to 1 over Michelson's fringes in the matter of light, which means that about 1000 times as many stars could be dealt with by the spectrum bands.

There is a similar advantage in regard to achromatism. It would be almost impossible to secure perfect achromatism when the base-line was very long, and an almost perfect achromatism is needed to obtain good results from Michelson's fringes. But in the case of the spectrum bands it is quite unnecessary that the system should be perfectly achromatic. Indeed, for several purposes it is even desirable that it should not be so. The exact degree and manner in which there is departure from achromatism

can be known from the adjustment of w_1 and w_2 and the refraction and dispersion of the atmosphere at the time; but for the evaluation of stellar diameters in the manner indicated it is quite unnecessary that these should be known. Nothing more is needed than that there should be interference bands and that these should not be too numerous.

We will now discuss the determination of parallax, for which also these bands should prove of service. For parallax we shall suppose that two pairs of siderostats are set up: a siderostat $A_1'B_1'$ side by side with A_1B_1, and another $A_2'B_2'$ side by side with A_2B_2. Let P_1, P_2 be the points where the reflecting surfaces of B_1, B_2 cut the common axis of the siderostats A_1B_1 and A_2B_2, and let P_1', P_2' be the points where the reflecting surfaces of B_1', B_2' cut the common axis of $A_1'B_1'$ and $A_2'B_2'$. Then we shall suppose that the points P_1, P_2, P_1', P_2' remain unchanged for all angular adjustments of the respective mirrors, and that the line joining P_1, P_2 is exactly perpendicular to the polar axes, and similarly for the line joining P_1', P_2'. It might not be easy to satisfy these conditions exactly in practice; but the apparatus can be calibrated to show the departure from them for any adjustment and equivalent corrections introduced in the calculations. We shall assume that these conditions are exactly satisfied and that the distances between P_1P_2 and between $P_1'P_2'$ are each equal to D.

The pair of siderostats, A_1B_1, A_2B_2, which, with their subsidiary apparatus, we may call σ, has that system of mirrors, wedges, telescope, etc. which we have already described. The other pair of siderostats, which we may similarly call σ', has a corresponding set a', d', e', v_1', v_2', w_1', w_2', L', b, c. b and c are common to σ and σ', these two mirrors extending completely across the paths of both σ and σ'. But a', d', w_1', w_2', and the telescope L' are independent for σ', and the adjustments of a', d', w_1', w_2' will in general be different from the adjustments of a, d, w_1, w_2. It is the difference between these adjustments which will serve for the determination of parallax.

Let us now consider two near stars, S of declination θ, and S' of declination θ'. σ and σ' are adjusted so that the beams from S travel along the axis of σ, coming together in L with equality of path-lengths, while the beams from

S′ travel along the axis of σ', and come together in L′ with equality of path-lengths. For simplicity we are disregarding refraction and aberration. If, now, we consider the two rays from S which pass through P_1 and P_2, these rays, considered at these two points, have a relative retardation of D sin θ, and since they meet with equality of path-lengths in L, there must be an optical difference of path-lengths, as measured from P_1, P_2 to L, equal to $D \sin\theta$. Similarly in the case of S′ and σ', there is an optical difference of path-lengths, as measured from P_1', P_2' to L′, equal to $D \sin\theta'$. But the difference between these two differences, and therefore between $D \sin\theta$ and $D \sin\theta'$, is made up by the difference in the adjustments of a, d, and a', d', and of w_1, w_2 and w_1', w_2', and can be evaluated, if desired, to nearly a hundredth part of a wave-length. We have therefore $D(\sin\theta - \sin\theta')$ with very great accuracy, and the adjustment of a, d, w_1, w_2 necessary to give equality of path-lengths for S, as compared with some standard adjustment, and confirmed by similar observations on stars of different altitudes, will give the necessary correction for refraction with very great accuracy. If it proved possible to use a sufficiently long base-line, we should have $\sin\theta - \sin\theta'$ to beyond $\frac{1}{100{,}000}$ of a second of arc, assuming of course that the stars we are dealing with are beyond the limit of resolution for the base-line used. By taking similar observations at different seasons and on a number of stars we shall have the north-south component of parallax to a similar degree of accuracy. This, of course, will be combined with the north-south component of the proper motion, but can be easily separated from it; and, knowing the north-south component of the parallax, we know the parallax.

We have assumed, however, a perfectly calm atmosphere, and have still to consider the effect of atmospheric disturbances, through which any given adjustment of σ, σ' only establishes a path equality for S, S′ for a single instant. We desire to find correct mean values for these adjustments, or, rather, for the difference between the adjustments, for it is with $D \sin\theta - D \sin\theta'$ that we are concerned rather than with $D \sin\theta$ and $D \sin\theta'$ separately. It is here that the spectrum bands will prove of service. For let us set w_1, w_2 so that if, for a state of the

atmosphere which is believed to represent the true mean of the disturbances, there is equality of path-lengths for some one wave-length λ of the star S, and at the same time a retardation of $\frac{\nu}{2}$ at some other wave-length ν. Then there will be a different configuration of the bands for every different retardation of one beam on the other. The configuration at any one instant will mark the exact retardation (for λ) and will show also which beam is in retardation. A similar adjustment is made for σ', S'.

Let us now imagine that two observers watch the spectra of S and S' respectively for a few minutes as the stars are crossing the meridian, during which time the chromatic conditions will not vary appreciably. A certain configuration of bands will indicate a retardation $\kappa\lambda$ for wave-length λ of the beam from B_2 with respect to that from B_1, and another configuration will indicate a retardation in the opposite sense $-\kappa\lambda$, κ being a selected small number, 3 or 4 perhaps. The observer counts the number of times in which the two configurations present themselves, say N_1 and N_2 respectively. The other observer does similarly during the same time for σ', S', obtaining N_1', N_2'. If the adjustments for path-length equality for λ represent true means, we shall have $N_1 = N_2$ and also $N_1' = N_2'$ if the average is taken over a large enough number of cases. But even if the adjustments for path-length equality for λ do not represent true means, yet if the *relative* adjustments of σ and σ' are correct, which is what we are really concerned with, since they give $\sin\theta - \sin\theta'$, we shall have $N_1 : N_2 - N_1' : N_2'$. If we obtain this relation over a wide enough average we may measure the difference of path represented by the relative adjustments of a, d ; a', d' and of w_1, w_2 ; w_1', w_2', and set this difference equal to $D(\sin\theta - \sin\theta')$. If the relation is not quite satisfied there will be a correction to which we can approximate.

This assumes that there are not large movements in the atmosphere. To deal with such one of the observers should be able to control b, c, a rotation of b, c affecting equally σ and σ', and counteracting broader atmospheric effects. When a configuration of the spectrum bands showed that one beam had a retardation upon the other exceeding a certain definite amount, say 6 or 7 wave-lengths, the observer would impose a correcting motion on b, c. This

motion might be one of invariable magnitude, so as to avoid the personal factor. The control might also be interchanged between the observers from time to time.

A different, and, with good seeing, possibly a better method would be the following. b, c do not extend across σ', which has separate mirrors b', c'. b, c are parallel to one another, and likewise b', c'. A parallel beam of light is thrown obliquely upon b, so as to be reflected to c', and then focussed by a telescope—or, better, after c', reflected by a fixed mirror back, nearly, but not quite, along its former path, into the telescope from which it issued — forming a spot of light, O. When there is any angular difference between b, c and b', c', O is displaced. Two observers, by controlling b, c and b', c', now endeavour to hold certain spectrum interference bands of S and S′ at positions which represent path-equality. In the process, O shifts back and forwards, and if it is received on a photographic plate, the centre, or position of maximum intensity of the track made by O, will indicate the *average* angular difference between b, c and b', c', which, of course, represents a difference in path-length to be added to that due to a, d and a', d'. The personal factor can be eliminated by photographing portion of the spectra at the same time. This can be done by using the blue end of the spectra for photography and the other end for vision, or by photographing spectra of the complementary beams found. The exact positions of the centres or maxima in the averaged spectra bands in the photographs, together with that of the O track, should supply data giving a result to very nearly a hundredth part of a fringe. A single observer could probably control both σ and σ', if the images of the two spectra were brought by mirrors into the same field of vision. It would be an advantage to have several pairs of siderostats, with a suitable number of observers, and thus to deal with a number of stars at the same time.

The apparatus suggested seems, so far as one can judge from theoretical considerations, to afford a means of measuring both the angular diameters of stars and their parallaxes with an accuracy in suitable cases approaching one-millionth of a second of arc. Since the knowledge of the parallax and the angular diameter together gives the absolute diameter, the apparatus would actually measure

the stars ; and one might perhaps suggest the name of *siderometer* for it, confining that of interferometer, as would seem convenient, to an instrument designed to measure wave-lengths. It may be remarked that the increase of accuracy which interference methods should give to a large body of astronomical data ought to be quite comparable to the increase of accuracy which interference methods brought to the measurement of wave-lengths, that is, an increase of 100 to 1000 times.

The apparatus described, with a north-south arrangement of siderostats, could, however, only increase the accuracy of the data of position ; but a similar refinement might be introduced into time measurements, as determined or tested by the passages of stars across the meridian, by an east-west arrangement of two siderostats. Mirrors would be placed to conduct the beams of any star, which both siderostats reflected along their respective axes, to ultimate union in a telescope by means of a half-silvered plate, the arrangements of the mirrors being such that there was equality of path-lengths as measured from the siderostats to the telescope. As an irresoluble star crossed the meridian there would now be equality of path-lengths, as measured from any arbitrary wave-front in space to the telescope ; and an observer, if his eye were sensitive to a sufficiently short time interval, would suddenly see a maximum, twice the average brightness, flash out, accompanied by dark minima before and after it. The duration of this maximum would, however, be far too brief in general to be observed by the eye.

In the case of a sufficiently bright star the maximum might be recorded by photo-electric apparatus. It could also be made accessible to the eye by means of some device —such as a moving mirror—which would continuously change the path-length of one of the beams at an appropriate rate for a few seconds as the star was crossing the meridian. This would incidentally afford a means of determining the east-west diameters of stars. But the east-west method as a whole would require, at the same time as it would supply a test for, a great increase in the accuracy of astronomical clocks.

LXXIX. *On an Application of the New Methods of the Calculus of Variations to some Problems in the Theory of Elasticity.* *By* N. M. Basu, *D.Sc.**

1. MOST problems of the mechanics of continuous media reduce ultimately to the solution of boundary-value problems associated with certain partial differential equations. The solution of such partial differential equations satisfying given boundary conditions can be effected in only a limited number of cases for which the boundaries are very simple curves. The success of the solution depends in general on the knowledge *á priori* of a complete set of particular solutions of the differential equation in question and the expansibility of an arbitrary function in terms of these functions in the manner of Fourier's Series. In 1908 Walter Ritz † proposed a new method which is applicable to a wider class of problems, and by which numerical values can in general be obtained to any desired degree of approximation in a comparatively simple manner. His method consists in treating the problem as one of the Calculus of Variations, into which it can easily be translated, since the differential equations of most mechanical problems can be derived from Hamilton's Principle. The problem is thus reduced to the determination of a function which, while satisfying certain boundary conditions and conditions of differentiability, renders a certain Integral expression minimum. But although Ritz's method gives a solution where the classical method fails, it does not afford any means of testing the measure of exactness realized in any particular case, and one has to be satisfied that the error must be small when further approximations do not give any appreciable variation in the value of the Integral. Recently, however, a method has been indicated by K. Friedrichs ‡ by means of which in certain cases it may be possible to associate with the given problem another Integral of which the maximum is exactly equal to the minimum of the given Integral, so that by considering the two extremum-problems we have a means of estimating the error involved in the solution obtained.

In the present paper the deflexion of a uniformly loaded square membrane has been worked out by Ritz's method, and the result tested by the consideration of an associated

* Communicated by the Author.

† See Crelle's *Journal*, Bd. cxxxv. pp. 1–61.

‡ See *Göttinger Nachrichten*, 1929.

problem set up by elementary reasoning. The result seems to be very satisfactory, inasmuch as a second approximation shows the error to be less than 1 in 400. The twisting couple of an elastic prism with square section, when subjected to torsion, has also been calculated, and compared with the result obtained by Saint Venant.

The Problem.

2. We know that the equilibrium position of a loaded membrane is determined by the equations

$$T\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+p=0 \quad \text{at inner points,} \quad . \quad . \quad (1)$$

$$u=0 \quad \text{at the boundary,} \quad . \quad (2)$$

where u is the vertical displacement at the point (x, y), T is the uniform tension with which the membrane is stretched, and $p(x, y)$ is the load per unit area at the point (x, y). If we write $\frac{p}{2T}u$ for u in the above equations, they reduce, when p is constant, to

$$\nabla_1^2 u+2=0 \quad \text{at all inner points,} \quad . \quad . \quad . \quad (3)$$

$$u=0 \quad \text{at the boundary} \quad . \quad . \quad . \quad . \quad (4)$$

It is now easily seen that the problem under consideration is equivalent to the following problem of the Calculus of Variations (*Problem I.*):

> *"Of all functions u which vanish on the boundary and which together with their first and second derivatives are finite and continuous at all inner points, to determine that one which makes the Integral*
>
> $$I[u]=\tfrac{1}{2}\iint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2-4u\right]dx\,dy \quad . \quad . \quad (5)$$
>
> *a minimum."*

The integral extends over the whole of the area under consideration.

That the minimum-problem enunciated above is *not meaningless* is easily seen, for certainly there are functions of the specified class for which $I[u]$ exists, and secondly $I[u]$ is bounded below. To prove this second point we note that by virtue of Schwarz's inequality

$$\left(\iint u\,dx\,dy\right)^2 \leq A\iint u^2 dx\,dy,$$

where A is the area of the region under consideration.

Further, as is well known *, since u vanishes on the boundary

$$\iint u^2 dx\,dy \leq c \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy,$$

where c is a positive constant. Thus we have

$$\left|\iint u\,dx\,dy\right| \leq c' \sqrt{\iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy},$$

from which the truth of the above assertion follows at once, c' being a positive constant.

3. We now consider a little more general problem, and shall come back to our original problem later. We consider the following problem (*Problem II.*):

> "*Of all functions $u(x, y)$, having a given value $\bar{u}$ on the boundary which together with their first and second derivatives are finite and continuous at all inner points, to find that one which makes the Integral*

$$M[u] \equiv \tfrac{1}{2} \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 4uf(x, y)\right] dx\,dy \quad . \quad (6)$$

> *a minimum, $f(x, y)$ being a given function.*"

It is easily seen that under suitable assumptions in regard to the functions u and f, the above problem has a sense. The Eulerian differential equation of this problem is obviously

$$\nabla_1^2 u + 2f(x, y) = 0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let us write

$$\frac{\partial u}{\partial x} = \xi, \qquad \frac{\partial u}{\partial y} = \eta.$$

If p and q are any two functions of x and y, we have

$$\left[\tfrac{1}{2}(\xi^2 + \eta^2) - 2uf\right] + \left[\tfrac{1}{2}(p^2 + q^2) - p\xi - q\eta + 2uf\right]$$
$$= \tfrac{1}{2}\left[(\xi - p)^2 + (\eta - q)^2\right] \geq 0,$$

for any values of ξ, η, u, p, and q, the sign of equality holding only when

$$\xi = p \quad \text{and} \quad \eta = q$$

simultaneously. Reintroducing the values of ξ and η, integrating over the area under consideration—p and q must,

* See, for instance, Courant, "Über die Eigenwerte bei den Differentialgleichungen der Mathematischen Physik," *Math. Zsft.* vii. pp. 13–16.

of course, be integrable functions—and partially transforming one of the area integrals into a boundary integral, we have

$$\mathrm{M}[u]+\tfrac{1}{2}\int\int(p^2+q^2)\,dx\,dy-\int(lp+mq)\bar{u}\,ds + \iint\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+2f\right)u\,dx\,dy\geq 0,$$

where ds is an elementary arc of the boundary and (l, m) are the usual direction-cosines of the outward-drawn normal at ds.

Since p and q are arbitrary functions, we can impose any conditions on them at pleasure. We make them satisfy the condition

$$\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+2f(x, y)=0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then we have

$$-\tfrac{1}{2}\iint(p^2+q^2)\,dx\,dy+\int(lp+mq)\bar{u}\,ds\leq \mathrm{M}[u]$$

for all admissible functions p, q, and u. If Problem II. has a solution $v(x, y)$ for which $\mathrm{M}[v]=d$, then, taking $u=v$ and writing

$$\mathrm{N}[p, q]\equiv-\tfrac{1}{2}\iint(p^2+q^2)\,dx\,dy+\int(lp+mq)\bar{u}\,ds,$$

we must have

$$\mathrm{N}[p, q]\leq d$$

for all functions p, q satisfying the equation (8). But since v satisfies the equation (7), $p=\dfrac{\partial v}{\partial x}$, $q=\dfrac{\partial v}{\partial y}$ satisfy the equation (8). Moreover, for these values of p and q the sign of equality must hold, as we have already remarked, so that we have

$$\mathrm{N}\left[\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right]=d=\mathrm{M}[v].$$

Thus we see that when the Problem II. has a solution, the expression $\mathrm{N}[p, q]$ does indeed attain its maximum value, and we have

$$\text{Max. } \mathrm{N}[p, q]=d=\text{Min. } \mathrm{M}[u].$$

We thus associate with Problem II. a new problem, namely, that of finding the maximum value of $\mathrm{N}[p, q]$, where p, q are subject to the condition (8), and we have, for all admissible p, q and u,

$$\mathrm{N}[p, q]\leq d\leq \mathrm{M}[u],$$

which furnishes the means for determining the degree of exactness to which the calculated value of $M[u]$ approximates its true minimum value.

To satisfy the equation (8) we proceed as follows:—

We notice that the condition may be written as

$$\frac{\partial}{\partial x}(p+F)+\frac{\partial q}{\partial y}=0,$$

where

$$F(x, y)=2\int f(x, y)\,dx.$$

We must therefore have

$$p+F=\frac{\partial w}{\partial y}$$

$$q=-\frac{\partial w}{\partial x}$$

where w is an undetermined function. Substituting these expressions for p and q in $N(p, q)$, the associated problem is defined in terms of only one unknown function w, which, on account of its connexion with the solution of Problem II., must possess continuous first and second derivatives in the region under consideration.

4. Returning now to our original problem (Problem I.), we see that it is only a special case of Problem II. with $\bar{u}=0$ and $f(x, y)=1$. Thus the associated problem becomes:

"*To find the maximum value of the integral*

$$J[w]=-\tfrac{1}{2}\iint\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2-4x\frac{\partial w}{\partial y}+4x^2\right]dx\,dy \quad (9)$$

where the admissible functions w are such as are continuous and have continuous first and second derivatives in the region under consideration."

Taking the sides of the square to be given by the equations $x=\pm 1$, $y=\pm 1$, we have

$$J[w]=-\frac{8}{3}+L[w], \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where

$$L[w]=-\tfrac{1}{2}\iint\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2-4x\frac{\partial w}{\partial y}\right]dx\,dy. \quad (11)$$

The problem therefore resolves itself into finding the maximum value of $L[w]$.

For the purpose of numerical calculations we note that if $\psi_1, \psi_2 \ldots . \psi_n, \ldots .$ are given functions, all vanishing on

the boundary, and $a_1, a_2, \ldots$ are arbitrary constants, and if u_n be that function of the type $a_1\psi_1 + a_2\psi_2 + \ldots + a_n\psi_n$ (which we shall hereafter call a function of class C_n) which makes $I[u]$ a minimum, then this minimum value $I[u_n]$ is given by

$$I[u_n] = -\iint u_n \, dx \, dy. \quad . \quad . \quad . \quad . \quad (12)$$

This easily follows from the consideration that the first variation of $I[u_n]$, namely

$$\iint \left[\frac{\partial u_n}{\partial x} \frac{\partial}{\partial x}(\delta u_n) + \frac{\partial u_n}{\partial y} \frac{\partial}{\partial y}(\delta u_n) - 2\delta u_n \right] dx\,dy, \quad (13)$$

must be zero, where δu_n, the small variation of u_n, is such that $u_n + \delta u_n$ also belongs to the class C_n. This is obviously satisfied by taking $\delta u_n = \epsilon . u_n$, where ϵ is a small constant. Substituting this value of δu_n in (13), and equating the result to zero, we derive the equation (12). In a similar manner, if w_n be that function of class C'_n* which makes $L[w]$ a maximum, then this maximum is given by

$$L[w_n] = \iint x \frac{\partial w_n}{\partial y} dx \, dy. \quad . \quad . \quad . \quad . \quad (14)$$

Choice of the ψ-functions for $I[u]$.

5. We know from a theorem due to Weierstrass that an arbitrary function which, together with its derivatives up to a given order, is finite and continuous, can always be represented by a series of polynomials which converge uniformly and can be differentiated the required number of times. Having regard to the form of the boundary and the conditions to be satisfied by the ψ-functions employed in Ritz's method †, we choose them to be given by

$$\psi_{mn} = (1-x^2)(1-y^2)x^m y^n, \; m, n = 0, 1, 2, \ldots.$$

These functions satisfy all the required conditions.

The Integral $I[u]$.

6. The nature of the problem shows that the solution must be symmetrical in x and y and contain only their even powers. We therefore give to m, n the values 0, 2, 4, ...

* We call this class $C'n$ instead of C_n as in general we shall use different ψ-functions for the integral $L[w]$.

† *Loc. cit.* p. 10. The conditions to be satisfied for the problem under consideration are slightly different—*e. g.*, the boundary condition $\frac{\partial \psi}{\partial n} = 0$ is no longer required, and the "Hauptableitungen" include only derivatives up to the second order.

Thus, as a first approximation we take

$$u_1 = a\psi_{00} = a(1-x^2)(1-y^2).$$

Substituting this value of u in $\mathrm{I}[u]$, and putting $\frac{\partial \mathrm{I}}{\partial a} = 0$, we get $a = \frac{5}{8}$.

Thus, as a first approximation we get

$$u_1 = \frac{5}{8}(1-x^2)(1-y^2), \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and from (12)

$$\mathrm{I}[u_1] = -\iint u_1 dx\, dy = -\frac{10}{9}. \quad . \quad . \quad . \quad (16)$$

As a second approximation we take, remembering that the solution must be symmetrical in x and y,

$$u_2 = a\psi_{00} + b(\psi_{02} + \psi_{20}) + c\psi_{22}.$$

Substituting $u = u_2$ in $\mathrm{I}[u]$, evaluating the integrals and putting

$$\frac{\partial \mathrm{I}}{\partial a} = \frac{\partial \mathrm{I}}{\partial b} = \frac{\partial \mathrm{I}}{\partial c} = 0,$$

we get the following three equations:

$$\begin{aligned} 280a + \quad 96b + \quad 8c - \ 175 &= 0. \\ 2016a + 2464b + \ 295c - 1470 &= 0. \\ 168a + \ 295b + 2788c - \ 147 &= 0. \end{aligned}$$

Solving these equations, and retaining only five places of decimals, we obtain as a second approximation

$$u_2 = (1-x^2)(1-y^2)[\cdot 58453 + \cdot 11733(x^2+y^2) + \cdot 00846x^2y^2] \ . \quad (17)$$

and

$$\mathrm{I}[u_2] = -252\cdot 72016/225. \quad . \quad . \quad . \quad . \quad (18)$$

We observe that the coefficients diminish rapidly, indicating rapid convergence.

The Associated Integral $\mathrm{J}[w]$.

7. We now determine the approximations to the maximum value of the associated integral. We observe that w is not subject to any other condition except the conditions of continuity and differentiability two times, but on forming the Eulerian differential equation for this variation problem it is found to be

$$\nabla_1^2 w = 0.$$

We therefore take * for the functions ψ, in this case, the plane harmonic polynomials, namely, 1, x, y, x^2-y^2, xy, x^3-3xy^2, $3x^2y-y^3$, etc. Simple considerations, into the details of which we need not enter here, show that the constant term and harmonics of odd degree, as well as those harmonics of even degree which contain only even powers of the variables, drop out of consideration. We therefore have to consider functions of the type

$$w=axy+b(x^3y-xy^3)+c(3x^5y-10x^3y^3+3xy^5)+\ldots.$$

As a first approximation we take

$$w=w_1=axy.$$

Substituting this value in $\mathrm{L}[w]$, and putting $\frac{\partial \mathrm{L}}{\partial a}=0$, we get $a=1$; so that

$$\left.\begin{aligned} w_1&=xy,\\ \mathrm{L}[w_1]&=\frac{4}{3},\end{aligned}\right\}\quad\ldots\ldots\quad(19)$$

and from (10)

$$\mathrm{J}[w_1]=-\frac{8}{3}+\frac{4}{3}=-\frac{4}{3}.\quad\ldots\ldots\quad(20)$$

As a second approximation we take

$$w=w_2=a'xy+b(x^3y-xy^3).$$

We then find

$$\left.\begin{aligned} w_2&=xy-\frac{7}{18}(x^3y-xy^3),\\ \mathrm{L}[w_2]&=\frac{208}{135},\end{aligned}\right\}\quad\ldots\quad(21)$$

and

$$\mathrm{J}[w_2]=-\frac{8}{3}+\frac{208}{135}=-\frac{152}{135}.\quad\ldots\quad(22)$$

The Estimation of the Error.

8. If d denotes the true minimum value of $\mathrm{I}[u]$, then, in accordance with the conclusions of Art. 3, we must have for every u and w

$$\mathrm{J}[w]\leq d\leq \mathrm{I}[u].$$

* We are here concerned only with finding approximations to the maximum value of $\mathrm{J}[w]$, and not with w itself, and it is therefore not necessary to examine whether the functions ψ used here satisfy the conditions required in Ritz's method. All that is required is that $\mathrm{J}[w]$ should have an upper limit, which is obviously true in the present case, since $\mathrm{J}[w]<0$, and then whatever function may be substituted for w we shall get a value not greater than this upper limit.

Using the first approximate solution u_1, we get

$$-\frac{152}{135} \leq d \leq -\frac{10}{9}.$$

Hence, if δ_1 denotes the error (*i. e.* $-\frac{10}{9} - d$),

$$\left|\delta_1\right| \leq \frac{152}{135} - \frac{10}{9} = \frac{2}{135}.$$

Since

$$\left|d\right| \geq \frac{10}{9},$$

the error is less than $\left(\frac{2}{135} \div \frac{10}{9}\right)$ 100 per cent., that is less than $1\frac{1}{3}$ per cent.

Using the second approximate solution u_2, we have similarly

$$-\frac{152}{135} \leq d \leq -\frac{252{\cdot}72016}{225}$$

and . . . .

$$\left|\delta_2\right| \leq \frac{152}{135} - \frac{252{\cdot}72016}{225} = {\cdot}002725.$$

Hence the error is less than $\cdot02725 \times 9 = \cdot2453$ per cent., which is less than 1 in 400.

If the deviation of the value of the integral $\mathrm{I}[w_2]$ from the true minimum value of $\mathrm{I}[u]$ be taken as a measure of the degree of exactness of the approximation, we must come to the conclusion that the solution u_2 obtained above must be a very good approximation indeed. Thus for all practical purposes the deflexion of a square membrane subjected to a uniform load p per unit area and a uniform tension T is given by the simple formula

$$u = \frac{p}{2\mathrm{T}} u_2,$$

where u_2 is given by (17).

The Torsion Problem.

9. The results obtained above can be utilized in calculating the twisting couple of an elastic prism whose cross-section is a square and which is subjected to torsion about its axis. It is well known that the solution of the Torsion Problem consists in the determination of the *torsion-function* ϕ, and that

if ψ denotes the conjugate of the torsion-function then ψ satisfies the following conditions :—

$$\nabla_1^2\psi = 0 \text{ at all inner points,}$$

$$\psi = \tfrac{1}{2}(x^2+y^2) + \text{constant, on the boundary.}$$

Putting

$$\psi = u + \tfrac{1}{2}(x^2+y^2) + \text{constant,}$$

we see that u satisfies the equations (3) and (4), so that for all practical purposes u is given by the equation (17). If μ denotes the rigidity of the material, τ the twist, ϕ the torsion-function, and ψ its conjugate, the twisting couple M is given by*

$$\left.\begin{aligned} \mathrm{M} &= \mu\tau\iint\left(x^2+y^2+x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)dx\,dy \\ &= \mu\tau\iint\left(x^2+y^2-x\frac{\partial\psi}{\partial x}-y\frac{\partial\psi}{\partial y}\right)dx\,dy \\ &= -\mu\tau\iint\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)dx\,dy, \end{aligned}\right\} \quad . \quad (23)$$

whence, on effecting Green's transformation, and remembering that u vanishes on the boundary,

$$\mathrm{M} = 2\mu\tau\iint u\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For the first approximation we have

$$\mathrm{M}_1 = \frac{20}{9}\mu\tau = 2{\cdot}2222\mu\tau.$$

Saint Venant's† result is $2{\cdot}2492\mu\tau$. The difference is $\cdot 027\mu\tau$, *i.e.*, approximately 1·2 per cent.

Using the second approximation (u_2) we get

$$\left.\begin{aligned} \mathrm{M}_2 &= 2\mu\tau \times 252{\cdot}72016/225 \\ &= 2{\cdot}2464\mu\tau. \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

The difference is $\cdot 0028\mu\tau$, *i.e.*, approximately ·124 per cent., which is less than 1 in 800. Thus we see that, using the value u_2, we obtain very close agreement with Saint Venant's result. The deviation is negligible for all practical purposes.

It may be noted that even without knowing Saint Venant's result it is possible to obtain an estimate of the error in the calculated value of M. For this purpose we observe from the equations (12) and (24) that

$$\mathrm{M} = -2\mu\tau\mathrm{I}[u],$$

* See Love's 'Theory of Elasticity,' 4th ed. p. 312.

† *Ibid.* p. 324.

where u is the function which makes $I[u]$ minimum, and therefore with successive approximations the calculated value of M must increase and approach its true value, the error percentage being the same as for $I[u]$. Hence, in accordance with the result of Art. 8, the error in the value of M given by (25) is certainly less than 1 in 400.

10. From the results obtained above it is clear that the deflexion of a membrane or the twisting couple of a prism can be determined by the above method to a fairly high degree of approximation. In the case of the torsion problem, however, it must be remembered that in order to determine the displacement or the state of strain and stress at any point it is necessary to know the torsion-function itself, and not its conjugate function. The above method determines the conjugate function to any desired degree of approximation, but it is not possible to construct the torsion-function from it, as it does not satisfy the potential equation. It will, however, be shown in a subsequent paper how the method of the calculus of variations leads to a direct determination of the torsion-function.

LXXX. *On the Torsion Problem of the Theory of Elasticity.* *By* Dr. N. M. Basu *.

1. THE solution of the torsion problem of the Theory of Elasticity consists in the determination of a function $\phi(x, y)$, called the *torsion-function*, which at all inner points of a plane area T satisfies the potential equation

$$\nabla_1^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while at the boundary S has the outward normal derivative given by

$$\frac{\partial\phi}{\partial n} = ly - mx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

l, m being the direction-cosines of the outward-drawn normal.

The usual method of determination of ϕ consists in solving the boundary value problem for the conjugate function

* Communicated by the Author.

$\psi(x, y)$, *i. e.*, a function which at all inner points of T satisfies the equation

$$\nabla_1^2\psi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while at the boundary S

$$\psi = \tfrac{1}{2}(x^2+y^2) + \text{a constant}. \quad . \quad . \quad . \quad . \quad (4)$$

Although the existence of the solution in all cases where the boundary is suitable is known from the usual Potential Theory, it is not an easy matter to construct the solution except when the boundary consists of one or more very simple curves. In the present paper it is proposed to give a general method of solving the problem which is applicable to a wide class of problems, and is specially suitable for numerical computations.

2. Let the area under consideration be a singly-connected region T, of which the boundary S is an analytic curve possessing continuous tangent and curvature. In these circumstances we are assured by the usual potential theory of the existence of a regular potential function ψ satisfying the equations (3) and (4), and therefore also of the torsion-function ϕ satisfying the equations (1) and (2). It is obvious that the solution contains an arbitrary constant and is otherwise unique. Since, however, the addition of an arbitrary constant implies a rigid body displacement, we shall suppose ϕ uniquely determined by adjusting the constant in such a manner that ϕ vanishes at a given point, say the origin of coordinates. Let $\phi = u(x, y)$ denote this unique solution of the boundary value problem for ϕ. We now consider the following problem in the calculus of variations :

> "*Of all functions w vanishing at the origin which together with their first derivatives are continuous in* T + S *and have continuous or piecewise continuous second derivatives in* T *to determine that one which makes the Integral*
>
> $$\mathrm{I}[w] \equiv \iint_{\mathrm{T}} \left[\left(\frac{\partial w}{\partial x} - y\right)^2 + \left(\frac{\partial w}{\partial y} + x\right)^2\right] dx\, dy$$
>
> *taken over the area* T, *a minimum.*"

It is clear that there are functions of the specified class for which $\mathrm{I}[w]$ exists and that $\mathrm{I}[w]$ is bounded below, so that the minimum problem formulated above has a sense. We shall first show that if the problem has a solution this solution must be unique. For, if possible, let v_1 and v_2 be two independent solutions each furnishing the same minimum value, say d, so that

$$\mathrm{I}[v_1] = \mathrm{I}[v_2] = d.$$

Writing $v_1 - v_2 = \zeta$, we have

$$\begin{aligned} I[v_1] &= I[v_2 + \zeta] \\ &= I[v_2] + D[\zeta] + 2K[v_2, \zeta], \end{aligned}$$

where

$$D[\zeta] = \iint_T \left[\left(\frac{\partial \zeta}{\partial x} \right)^2 + \left(\frac{\partial \zeta}{\partial y} \right)^2 \right] dx\, dy$$

and

$$K[v_2, \zeta] = \iint_T \left[\left(\frac{\partial v_2}{\partial x} - y \right) \frac{\partial \zeta}{\partial x} + \left(\frac{\partial v_2}{\partial y} + x \right) \frac{\partial \zeta}{\partial y} \right] dx\, dy.$$

Since v_1 and v_2 belong to the class of admissible functions, ζ also belongs to the same class. Hence, in accordance with well-known principles of the calculus of variations,

$$K[v, \zeta] = 0.$$

Thus we must have

$$D[\zeta] = 0,$$

whence, on account of the continuity of ζ and its first derivatives, ζ must be a constant, which, again, must be zero, as ζ is zero at the origin. Thus *the solution, if it exists, must be unique.*

We shall now show that the function u mentioned above does actually solve the minimum-problem. Let us consider any admissible function w, and let $w - u = \eta$. Then, in accordance with the notation used above,

$$I[w] = I[u] + D[\eta] + 2K[u, \eta].$$

Using Green's transformation formula, we have

$$\begin{aligned} K[u, \eta] &= -\iint_T \eta \nabla_1{}^2 u\, dx\, dy + \int_S \eta \left(\frac{\partial u}{\partial n} - ly + mx \right) ds \\ &= 0, \end{aligned}$$

since u is the torsion-function and satisfies the equations (1) and (2). Thus

$$I[w] \geqq I[u],$$

and the sign of equality holds only when

$$D[\eta] = 0,$$

i. e., when

$$\eta = \text{a constant} = 0,$$

since η must be 0 at the origin. We thus conclude that *the solutions of both problems exist and that they are identical.*

3. We shall now show that for the class of problems under consideration the torsion-function u, together with its first derivatives, can be approximated as closely as we please in the closed region T + S by means of a series in the harmonic polynomials

$$1,\ x,\ y,\ x^2-y^2,\ xy,\ \ldots$$

If we denote the conjugate function of u by v, then

$$u+iv=f(z),$$

where $f(z)$ stands for a regular analytic function of the complex variable z in the region T. Now, since on S $v=\frac{1}{2}(x^2+y^2)+$a constant, v is analytic on the boundary S, which is an analytic curve and is a regular potential function inside the region T. Hence v can be analytically continued * over the boundary S, that is, there exists a regular potential function V in a region T′ of which T + S is a partial region, and such that $V=v$ in T + S. It follows that there exists a regular analytic function $F(z)$ in T′ such that $F(z)=f(z)$ in T + S. Now, in accordance with Runge's Theorem † we can approximate $F(z)$ in every partial region of T′, and therefore in T + S, as closely as we please by means of a polynomial in z. Since $u(x, y)$ is the real part of $F(z)$ in T + S, the assertion made above is established, since the harmonic polynomials are nothing but the real and imaginary parts of powers of z.

4. We now restrict the admissible functions w to only such functions as are regular potential functions in T, and can, together with their first derivatives, be approximated as closely as we please in T + S by a series of harmonic polynomials. This does not imply any real limitation, nor is the minimum value thereby altered, as the solution of the problem is now known to belong to this restricted class of functions.

Let $\psi_1, \psi_2, \psi_3, \psi_4, \ldots$ stand for the harmonic polynomials $x,\ y,\ x^2-y^2,\ xy\ \ldots$, the constant 1 being left out of consideration as w is to be zero at the origin. Let u_n be the nth approximation to u according to Runge's Theorem, so that u_n is of the form

$$u_n = c_1\psi_1 + c_2\psi_2 + \ \ldots\ + c_n\psi_n,$$

* See Goursat, ' Cours d'Analyse Mathématique,' tome iii.
† See Bieberbach, ' Funktionentheorie,' Bd. I.

where $c_1, c_2, \ldots c_n$ are constants. If w_n be that function of the class C_n† which makes $I[w]$ a minimum, then

$$I[u_n] \geqq I[w_n] \geqq d.$$

But since when n increases indefinitely u_n tends to u and the first derivatives of u_n tend to those of u, it follows that

$$\lim_{n=\infty} I[u_n] = I[u] = d.$$

Hence

$$\lim_{n=\infty} I[w_n] = d.$$

Thus $w_1, w_2, \ldots w_n, \ldots$ form a minimal sequence. If we can now show that the sequence $w_1, w_2, \ldots$ does tend to a limiting function w, which vanishes at the origin and which, together with its derivatives, is continuous in T and such that

$$I[w] = d,$$

then w will be the required solution.

5. To prove the convergence of the sequence we shall require the following convergence theorem :

> "*If* $\phi_1, \phi_2, \ldots$ *denote a sequence of regular potential functions having a given value at a fixed point in a given region* T, *such that*
>
> $$\lim_{\substack{m=\infty \\ n=\infty}} D[\phi_m - \phi_n] = 0,$$
>
> *then the sequence converges uniformly towards a regular potential function* ϕ, *and the partial derivatives of the sequence converge towards the corresponding partial derivatives of* ϕ *in every partial region* T* *of* T.

We prove the theorem in the following way. Without loss of generality we take the given value as 0 and the given point as the origin. Put

$$\frac{\partial \phi_m}{\partial x} = \theta_m, \quad \frac{\partial \phi_m}{\partial y} = \chi_m.$$

Then it follows that θ_m and χ_m are regular potential functions in T, and that

$$\lim_{\substack{m=\infty \\ n=\infty}} \iint (\theta_m - \theta_n)^2 dx\, dy = 0, \quad \lim_{\substack{m=\infty \\ n=\infty}} \iint (\chi_m - \chi_n)^2 dx\, dy = 0,$$

† A function of the type $a_1\psi_1 + a_2\psi_2 + \ldots + a_n\psi_n$ where, $a_1, a_2, \ldots a_n$ are constants, will be called a function of the class C_n.

the integrals being taken over T or any partial region of the same. Hence by a well-known theorem * the sequences θ_1, θ_2, ... and χ_1, χ_2, ... converge uniformly to regular potential functions, and so also do their partial derivatives. Let these limiting functions be denoted by p and q. Then $\frac{\partial \theta_m}{\partial y}$ and $\frac{\partial \chi_m}{\partial x}$ also converge uniformly to $\frac{\partial p}{\partial y}$ and $\frac{\partial q}{\partial x}$ respectively.

But

$$\frac{\partial \theta_m}{\partial y} - \frac{\partial \chi_m}{\partial x} = 0 \text{ for all } m.$$

Hence

$$\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = \lim\left(\frac{\partial \theta_m}{\partial y} - \frac{\partial \chi_m}{\partial x}\right) = 0.$$

Thus there exists a function ϕ such that

$$\frac{\partial \phi}{\partial x} = p = \lim_{m=\infty} \frac{\partial \phi_m}{\partial x},$$

and

$$\frac{\partial \phi}{\partial y} = q = \lim_{m=\infty} \frac{\partial \phi_m}{\partial y}.$$

Now

$$\lim \phi_m(\mathrm{P}) = \lim \int_0^{\mathrm{P}} \left(\frac{\partial \phi_m}{\partial x} dx + \frac{\partial \phi_m}{\partial y} dy\right)$$

$$= \int_0^{\mathrm{P}} \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy\right)$$

$$= \phi(\mathrm{P}),$$

$\phi(0)$ being chosen to be zero. It its easily seen that $\phi(\mathrm{P})$ is a potential function, and the convergence of the partial derivatives follow without difficulty. Thus the theorem is proved.

6. Since w_n is that function of the class C_n for which $I[w]$ has its minimum value, it follows by the usual principles of the calculus of variations and with the notations of Art. 2 that

$$K[w_n, \zeta_n] = 0$$

for every admissible function ζ_n of class C_n. Now, if $m < n$, $\zeta_n = w_n - w_m$ is obviously a function of the class C_n, hence

$$K[w_n, w_n - w_m] = 0.$$

* See Kellogg, 'Foundations of Potential Theory,' p. 268, Theorem X.

On the other hand,

$$\begin{aligned} I[w_m] &= I[w_n + w_m - w_n] \\ &= I[w_n] + D[w_m - w_n] + 2K[w_n, w_m - w_n] \\ &= I[w_n] + D[w_m - w_n]. \end{aligned}$$

But

$$\lim I[w_m] = \lim I[w_n] = d.$$

Hence

$$\lim D[w_m - w_n] = 0.$$

In accordance with the convergence theorem proved in the preceding article, it follows that the sequence $w_1, w_2, \ldots w_n, \ldots$ converges uniformly to a regular potential function w in every partial region T* of T. The same is also true in regard to the sequence of the derivatives which converges to the derivative of w. It then follows that

$$\lim_{n=\infty} I_{T^*}[w_n] = I_{T^*}[w].$$

If we define the integral over the region T as the limit of the integral over the region T* as T* approaches T as a limit, we have

$$I_T[w_n] \geqq I_{T^*}[w_n].$$

Therefore

$$\lim_{n=\infty} I_T[w_n] \geqq \lim I_{T^*}[w_n] = I_{T^*}[w] \text{ for every } T^*,$$

or

$$d = \lim I_T[w_n] \geqq I_T[w].$$

But since d is the minimum value of the integral, we must have

$$I_T[w] \geqq d$$

Hence

$$\lim I_T[w_n] = I_T[w] = d.$$

Thus the function w is a solution of the minimum problem, and, as explained in Art. 2, must be identical with the torsion-function u.

We have thus a method for constructing the torsion-function in a convergent infinite series of simple polynomials applicable to all cases where the region under consideration is of the kind considered in this paper.

7. If the boundary of the area, instead of being a single analytic curve, consists of a finite number of pieces of analytic curves cutting at definite angles, then the arguments of Art. 3 are not wholly applicable, for in this case the larger region T′ referred to there will not necessarily contain

the angular points as inner points. If, however, we assume that the torsion-function is such that it admits of analytical continuation also at these angular points, then the subsequent arguments hold *in toto*, and we can construct the torsion-function exactly as before. Having regard to the physical nature of the problem, it seems quite reasonable to make this assumption. To test the validity of the assumption and the power of the method detailed above the case of the prism with square section has been worked out, and the result compared with the known result due to Saint Venant, with which it is found to agree very closely. The details of the calculations are given in the following paragraph.

8. We shall find successive approximations to the function w which makes the integral $\mathrm{I}[w]$ defined in Art. 2 a minimum—that is, we shall construct the sequence $w_1, w_2, \ldots$. To avoid unnecessary troubles of calculation we shall make use of the results obtained in the previous paper *. For this purpose we put

$$w = xy - v(x, y).$$

Then we have $$\mathrm{I}[w] = -2\mathrm{J}[v],$$

where J is the integral defined in Art. 4 of the paper referred to above. Thus the problem is to find the function v which makes the integral $\mathrm{J}[v]$ a maximum, the coordinate functions being the plane harmonic polynomials $\psi_1, \psi_2, \psi_3, \ldots$ of Art. 4. To determine the nth approximation we assume

$$v = a_1\psi_1 + a_2\psi_2 + \ldots + a_n\psi_n,$$

and substituting in J, determine the values of $a_1, a_2, \ldots a_n$ which make J a maximum. It will be found that the equations fall into two groups, one of these groups being a group of homogeneous linear equations in those members of the constants $a_1, a_2, \ldots a_n$ which are associated with the polynomials of odd degree and those polynomials of even degree in which the variables x, y occur in their even powers. These constants are therefore 0. We have therefore to consider only the polynomials of the following type :—

$$xy,\ x^3y - xy^3,\ 3x^5y - 10x^3y^3 + 3xy^5,\ \ldots$$

As a first approximation we have already found in the paper mentioned

$$v = xy - \tfrac{7}{18}(x^3y - xy^3),$$

whence $$w_1 = \tfrac{7}{18}(x^3y - xy^3).$$

* "On an Application of the New Methods of the Calculus of Variations to some Problems in the Theory of Elasticity," *suprà*, p. 886.

To determine a second approximation we assume

$$v = axy + b(x^3y - xy^3) + c(3x^5y - 10x^3y^3 + 3xy^5) + d(x^7y - 7x^5y^3 + 7x^3y^5 - xy^7).$$

We find, retaining five decimal places,

$$a = 1,\ b = -\cdot 36255,\ c = 0,\ d = \cdot 02147.$$

Hence, as a second approximation we get

$$w_2 = \cdot 36255(x^3y - xy^3) - \cdot 02147(x^7y - 7x^5y^3 + 7x^3y^5 - xy^7).$$

Following the method of the paper mentioned, it is easy to show that, if w_n denotes the nth approximation for w, then

$$\mathrm{I}[w_n] = \iint \left[x^2 + y^2 + x\frac{\partial w_n}{\partial y} - y\frac{\partial w_n}{\partial x} \right] dx\, dy.$$

Hence if $\mathrm{M}[w_n]$ denotes the value of the twisting couple calculated with the nth approximation w_n, we have

$$\mathrm{M}[w_n] = \mu\tau \mathrm{I}[w_n] = -2\mu\tau \mathrm{J}[v_n],$$

so that, calculating for the second approximation, we get

$$\mathrm{M}[w_2] = 2\cdot 2494 \mu\tau.$$

Again, in the paper under reference it has been shown that the maximum value of $\mathrm{J}[v]$ is exactly equal to the minimum value of the integral (there denoted by $\mathrm{I}[u]$)

$$\mathrm{I}'[u] = \tfrac{1}{2}\iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 4u \right] dx\, dy,$$

and that the minimum value of $\mathrm{I}'[u]$ is less than or equal to

$$\mathrm{I}'[u_2] = -\frac{252\cdot 72016}{225} = -1\cdot 1232.$$

Thus if M denotes the true value of the twisting couple we must have

$$2\cdot 2494\mu\tau \geqq \mathrm{M} \geqq -2\mu\tau(-1\cdot 1232) = 2\cdot 2464\mu\tau.$$

This shows that the error in taking $2\ 2494\mu\tau$ as the value of the twisting couple as calculated with the second approximation w_2 does not exceed about 1 in 750. The deviation from Saint Venant's result ($2\cdot 2492\mu\tau$) is only $\cdot 0002\mu\tau$, which is about 1 in 12,000. The agreement is extraordinarily good.

We thus conclude that the expression w_2 obtained above may be used as the required torsion-function for all practical purposes. If greater exactness be required higher approximations may be calculated without difficulty.

LXXXI. *Early Developments in A.C. Circuit Theory. Some Notes on the Application of Complex Methods to the Solution of A.C. Circuit Problems.* *By* G. WINDRED *.

(1) *Neglect of the Theory in England.*

THE first application, in America more than thirty years ago, of the method of complex numbers to the solution of problems in alternating-current circuit theory was a highly important step—one which has, however, received very much less subsequent attention than is warranted by its extreme practical importance.

This circumstance applies particularly to England, which has contributed practically nothing to the furtherance of the subject, although, strangely enough, the names of many English mathematicians are encountered in the history of the purely mathematical applications of complex numbers †.

Relatively few English text-books of electrical theory contain any reference to the use of complex methods, and where such reference does occur the bare fundamentals are stated; their development and application are left to the reader's unpracticed imagination.

It must here be remarked that such extension of elementary principles is not so simple as is conveniently assumed in these text-books. The application of complex methods to subjects in which any degree of complexity is reached may be attended with considerable mathematical difficulty.

Under these conditions progress is of necessity slow, as is amply attested by the almost complete neglect of the subject in this country.

Perhaps the most notable exception to this general tendency of English text-books is Dr. Alexander Russell's 'Treatise on the Theory of Alternating Currents,' 2nd ed. Cambridge, i. (1914), in chapter vii. of which the theory is developed in sufficient mathematical detail to give a clear insight into the fundamentals of its application.

Some of the more advanced parts of the theory are touched upon in arts. 522, 537, and 583 *a* of Sir J. H. Jeans's 'Mathematical Theory of Electricity and Magnetism,' 5th ed. Cambridge (1927), which presupposes a previous

* Communicated by the Author.

† See the author's paper "History of the Theory of Imaginary and Complex Quantities," 'Mathematical Gazette,' xiv. pp. 533-541 (Oct. 1929).

reading of the theory of complex numbers, although an outline of the theory is given in arts. 309 *et seqq.*, introductory to conjugate functions and problems in two dimensions.

These last two subjects are somewhat outside the scope of the present inquiry, but a recent paper by Dr. F. W. Carter, making use thereof, has given rise to an interesting (even if somewhat heated) discussion which has a bearing on our present considerations. This paper, entitled "The Magnetic Field of the Dynamo-Electric Machine," Journ. I.E.E. lxiv. pp. 1115–1138, sets forth in an admirable manner the applications of the theory of conjugate functions to the solution of such two-dimensional problems as occur in connexion with the magnetic circuits of dynamo-electric machinery. In the discussion of this paper—which will well repay the careful attention of the student of mathematical physics—several papers and books of a fundamental nature are quoted. In general the discussion affords an excellent perspective of the confused state of the theory, and the misapprehension which exists concerning its first applications.

(2) *Systematic Use in America and Germany.*

Among American and German theorists full use has been made of the theory of complex numbers in its applications to electrical circuit theory, with a consequent simplification of the attendant calculations, together with a more elegant means of exposition.

It must be evident that more elegant methods of analysis inevitably tend to simplify operations, and it is further obvious that such methods are not attended with such risks of error as exist with the more cumbersome methods of working.

With reference to the introduction of vector analysis into electrical calculations, Oliver Heaviside, whose opinion on such matters is far above question, has written :—

"The notorious difficulty of understanding and working Quaternions will always be a bar to their serious practical use by any but mathematical experts. But, on the other hand, a vector algebra and analysis of a simple kind, independent of the quaternion, and readily understandable and workable, can with great advantage take the place of much of the usual cumbrous Cartesian investigations, and be made generally useful in all physical mathematics concerning vectors, and be employed, comparatively speaking, by the multitude"*.

* Page xi of the Preface to his 'Collected Papers,' i. (1892).

The general method of complex numbers is such a method as Heaviside here describes.

It is a significant fact that this accomplished investigator employed complex methods frequently in his calculations; his work is an example of mathematical exposition, and it is evident that he clearly recognized the scope afforded by such methods in the solution of many of his original and difficult problems.

It appears that Heaviside was the first to apply imaginary Bessel functions to the problem of oscillatory induced currents in cores. This application is found in §10 of a series of papers contributed to 'The Electrician' * in 1884 and 1885, in which the mathematical side of the problem is treated exhaustively. The subject has also received the attention of Lord Kelvin†, who introduced his "bei" and "ber" functions ‡ in this connexion.

Apart from these isolated examples of the use of complex methods in applied mathematics, we find little else of importance in English scientific literature. This applies in particular to electrical circuit theory, with which we are at present concerned; no effort has been made in this branch of electrical theory to obtain the benefits of complex representation.

In Germany the method has been freely adopted in electrical circuit theory, and has found its way into standard electrical text-books. Thus, in Dr. Adolf Thomälen's well-known treatise on 'Electrical Engineering' (Eng. trans. by A. W. O. Howe; Arnold) we find, on pp. 467 *et seqq.*, a lucid exposition suited to direct use in practical problems.

In America the development has been perhaps greater than in Germany, mainly on account of the efforts of Dr. Steinmetz in this direction. Practically all of his published works contain extensive applications of complex numbers, which, as we shall see later, he did much to develop in connexion with electrical circuit theory. In point of fact the present state of development of this part of the subject is almost entirely due to the efforts of Steinmetz, although it appears that he was not, as is generally supposed, responsible for the first inception of these methods. A true perspective of this development is afforded by a brief historical retrospect, to which we now proceed.

* 'The Electrician,' xiii. pp. 583 *et seqq.*

† Journ. I. E. E. xviii. p. 4 (Jan. 1889); also British Association Report, p. 736 (1890).

‡ See also Russell, 'Alternating Currents,' ch. xvi.

(3) *Analytical Methods of Oliver Heaviside.*

Oliver Heaviside's electrical papers abound with original mathematical methods. In view of the facility with which he devised and applied these methods it is to be expected that in the mathematical branches of electrical theory he was considerably ahead of his contemporaries. This fact is proved by the long period which elapsed before the proper appreciation of this part of his work.

Examination of his numerous papers makes it evident that Heaviside never failed to obtain a solution to his problem on account of unsuitable mathematical methods. In this respect he was quite independent of convention, but at the same time highly conscious of the necessity for mathematical rigour. Thus his works bear the mark of accurate analysis from logically founded principles.

We have previously remarked upon Heaviside's use of imaginary Bessel functions. In other instances he made extensive use of the properties of imaginary functions, but from the present viewpoint the most important piece of analysis is contained in part v.* of a series of papers contributed to the 'Philosophical Magazine' in the years 1886 and 1887, "On the Self-Induction of Wires."

The problem concerns the circuit conditions in a homogeneous line of length l, resistance R, and inductance L, acted upon at one end by an impressed force $V_0 \sin nt$.

With a current C in the line, and a potential-difference V at distance z from the transmission end of the line, the fundamental line-equations are

$$\left.\begin{aligned} -\frac{dC}{dz} &= \left(K + S\frac{d}{dt}\right)V, \\ -\frac{dV}{dz} &= R''C, \end{aligned}\right\} \quad \ldots \quad (1)$$

where K and S are the conductance of the insulator and the electric capacity respectively, and $R'' = R' + L'(d/dt)$ in the periodic case, R' and L' being the ultimate values of R and L at the given frequency.

If the terminal conditions be represented by

$$\left.\begin{aligned} V &= Z_1 C \text{ in the case where } z = l, \\ -V_0 \sin nt + V &= Z_0 C \text{ in the case where } z = 0, \end{aligned}\right\} \quad (2)$$

* Phil. Mag. p. 10 (Jan. 1887). Heaviside's operational methods have recently received considerable attention; see, *e.g.*, Ernst J. Berg, 'Heaviside's Operational Calculus,' New York, 1929. The theory we are at present using has also received the attention of Lord Rayleigh, *cf.* 'Theory of Sound,' Cambridge, i. 1894, arts. 235*p*, 235*u*, 235*x*.

then $V = Z_0 C$ is the terminal condition when $z = 0$ if there be no impressed force.

To obtain the solution Heaviside takes the equation *

$$C_1 = (\cos mz + q_1 \sin mz) \frac{\cos mz_2 + q_0 \sin mz_2}{(m/S'')(q_0 - q_1)} e_2,$$

of which the required solution is a special case. In the above equation $\cos mz$ and $\sin mz$ are the normal current-functions in a homogeneous line, neglecting variability of electrical constants †, m is the function of p given by $-m^2 = R''S''$, and q_0 and q_1 are respectively

$$q_0 = -\frac{S''}{m} Z_0,$$

$$q_1 = \frac{(m/S'') \sin ml - Z_1 \cos ml}{(m/S'') \cos ml + Z_1 \sin ml}.$$

If in the given equation we take

$$S'' = K + S\frac{d}{dt},$$

$$R'' = R' + L'\frac{d}{dt},$$

and put F^2 for $-m^2 = R''S'' = (K + Sp)(R' + L'p)$, we obtain, by taking $z_2 = 0$, $e_2 = V_0 \sin nt$, and putting the equation in the exponential form,

$$C = \frac{\left(\frac{F}{S''} + Z_1\right)\epsilon^{F(l-z)} + \left(\frac{F}{S''} - Z_1\right)\epsilon^{-F(l-z)}}{\epsilon^{Fl}\left(\frac{F}{S''} + Z_1\right)\left(\frac{F}{S''} - Z_0\right) - \epsilon^{-Fl}\left(\frac{F}{S''} - Z_1\right)\left(\frac{F}{S''} + Z_0\right)} V_0 \sin nt. \quad . \quad . \quad . \quad (5)$$

If now in F, S'', Z_0, and Z_1 we take $d^2/dt^2 = -n^2$, we can put

$$C = \frac{P' + Q'\frac{d}{dt}}{A' + B'\frac{d}{dt}} V_0 \sin nt =$$

$$\frac{(A'P' + B'Q'n^2) + (A'Q' - B'P')\frac{d}{dt}}{A'^2 + B'^2 n^2} V_0 \sin nt, \quad (6)$$

* Phil. Mag. p. 419 (Nov. 1886), *vide* equations (162 *b*).

† *Ibid.* equations (174), (160 *b*).

which is the complete expression for the amplitude and phase difference at any point. For the amplitude we have

$$C_0 = V_0(A'^2 + B'^2 n^2)^{-\frac{1}{2}}(P'^2 + Q'^2 n^2)^{-\frac{1}{2}}, \quad . \quad . \quad (7)$$

in which expression A' and B' are constant and P' and Q' are functions of z.

We come now to the most important part of this analysis from the present viewpoint. Heaviside puts

$$\left.\begin{aligned} F &= P + Qi, \\ Z_1 &= R_1' + L_1' ni, \\ -Z_0 &= R_0' + L_0' ni, \\ ni &= p \quad \text{and} \quad i = \sqrt{-1}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

where

and obtains

$$P = (\tfrac{1}{2})^{\frac{1}{2}}\{(R'^2 + L'^2 n^2)^{\frac{1}{2}}(K^2 + S^2 n^2)^{\frac{1}{2}} + (KR' - L'Sn^2)\}^{\frac{1}{2}},$$
$$Q = (\tfrac{1}{2})^{\frac{1}{2}}\{(R'^2 + L'^2 n^2)^{\frac{1}{2}}(K^2 + S^2 n^2) - (KR' - L'Sn^2)\}^{\frac{1}{2}}.$$

The complex form of the expressions in (8) for the impedances should be particularly noted, although it must be remarked that up to the end of the investigation we are now using Heaviside makes no direct application of this form of the impedance equation.

(4) *Early Papers of Dr. A. E. Kennelly.*

In the 'Transactions of the American Institute of Electrical Engineers' for April 1893, x. pp. 175–216, we find what is evidently the first application of complex quantities to problems in electrical technology, contained in a paper on "Impedance" by A. E. Kennelly. This paper, which was read on 18th April, 1893, is of fundamental importance on account of its treatment of practical problems by the method of complex quantities. It is also the first paper to give a consistent account of this method in its application to alternating-current circuit theory.

On page 186 of the paper Dr. Kennelly points out that all continuous-current corollaries from Ohm's law become applicable to harmonic currents when the admittances and impedances are treated vectorially, and that Kirchhoff's laws, while true of instantaneous conditions in alternating-current circuits, can only be applied when the currents are all in phase, and arithmetical operations with the circuit constants are permissible. In any other circumstances vectorial methods are necessary, and some form of vectorial algebra is required for the complete analysis of the circuit conditions.

Referring to these observations Dr. Kennelly makes the following important statement :—

"Any combination of resistances, non-ferric inductances, and capacities, carrying harmonically alternating currents, may be treated by the rules of unvarying currents, if the inductances are considered as resistances of the form $pl\sqrt{-1}$, and the capacities as resistances of the form $-\frac{1}{kp}\sqrt{-1}$, the algebraic operations being then performed according to the laws controlling 'complex quantities.'"

In the expressions mentioned, $p=\omega=2\pi f$, where f is the frequency, l the inductance, and k the capacity of the circuit. These forms are identical with modern practice ; they have been applied extensively by their inventor and by the late Dr. C. P. Steinmetz, whose work we consider hereafter.

A further paper by Dr. Kennelly, "On the Fall of Pressure in Long Distance Aternating Current Conductors," published in the 'Electrical World,' New York, Jan. 6th, 1894, xxiii. pp. 17–19, introduces the system of vectorial notation developed by this author, in which the disposition of an angle with respect to its components is indicated by one of the forms $\underline{/\theta}$, $\underline{\backslash\theta}$, $\overline{/\theta}$, $\overline{\backslash\theta}$. In electrical work this system has evident advantages over the algebraical forms, since it indicates at once the nature of the angular component of any expression.

The paper is concerned with the rigorous computation of the fall of pressure in alternating-current lines of known constants. The author makes use of the two fundamental relations

$$U = u\cosh Lz + yi\sinh Lz,$$

$$I = i\cosh Lz + \frac{u}{y}\sinh Lz,$$

which are based on Oliver Heaviside's solution* of the differential equations of an alternating current line, and which represent respectively the pressure and current at any point of the line, where u is the pressure at the receiving end, L is the distance between the supply end and the receiving end, i is the load current, and y and z are given by the expressions

$$y = \left[\frac{r+jl\omega}{g+jc\omega}\right]^{\frac{1}{2}}, \qquad z = [(r+jl\omega)(g+jc\omega)]^{\frac{1}{2}},$$

* *Vide* 'Electromagnetic Theory,' i. pp. 451–452 (1893).

where r = resistance of line per unit length,
l = inductance of line per unit length,
g = leakage conductance of line per unit length,
c = capacity of line per unit length,
$\omega = 2\pi f$,
and $j = \sqrt{-1}$.

The values of the necessary constants are worked by vectorial analysis, and applied to the solution of a typical transmission line problem with arbitrarily assigned circuit constants, a comparison being made between two specimen lines from which the effects of high pressure on the neutralizing electrostatic action are made evident.

In the foregoing investigation the forms of the expressions for y and z should be particularly noted. They represent a practical extension of the same author's preliminary statement which we have already quoted.

The practical application of complex quantities is further extended by Dr. Kennelly in his treatise on 'The Application of Hyperbolic Functions to Electrical Engineering Problems,' University of London Press, 3rd ed. (1925), in chapter v. of which the subject is exhaustively treated in its relation to the types of electrical problems with which this work is concerned.

(5) *Development by Dr. C. P. Steinmetz.*

Since the appearance of Dr. Kennelly's papers of 1893 and 1894 the subject has received considerable extension, notably, as we have already observed, in America, where the contributions of the late Dr. C. P. Steinmetz are the most outstanding in this connexion. His extensive paper on "Complex Quantities and their Use in Electrical Engineering," * before the International Electrical Congress of Chicago in August 1893, contains an extension of Dr. Kennelly's work, and gives, for the first time, a generalization of Kirchhoff's laws to the inclusion of alternating-current phenomena. The majority of Dr. Steinmetz's works † on alternating-current theory contain applications of complex methods, which are also to be found in many of his electrical papers.

* Proc. Int. El. Congress, A.I.E.E., pp. 33–74, also pp. 24, 25 (1894).

† A demonstration of complex quantities is contained in his 'Theoretical Elements of Electrical Engineering,' 3rd ed. New York, 1909, art. 14, pp. 83 *et seqq.* *Cf.* also 'Theory and Calculation of Alternating-Current Phenomena,' New York, 1897.

This circumstance has resulted in a general opinion that the inception of these methods is due to him, but, as we have shown above and as Dr. Kennelly has already pointed out *, their origin rests elsewhere.

The priority of Dr. Kennelly is acknowledged by Steinmetz himself in his discussion † of Dr. Kennelly's paper on "Impedance," wherein he remarks, with reference to the mathematical treatment of vector quantities :—"It is, however, the first instance here, so far as I know, that attention is drawn by Mr. Kennelly to the correspondence between the electrical term 'impedance' and the complex numbers." Steinmetz also adds :—"The importance hereof lies in the following : the analysis of the complex plane is very well worked out, hence, by reducing the electrical problems to the analysis of complex quantities they are brought within the scope of a known and well understood science."

At the hands of Steinmetz the method of complex quantities has received considerable development in its application to electrical engineering problems. The present state of development is, however, also due to many of his contemporaries, as well as more recent exponents of electrical theory, a circumstance which makes the bibliography of the subject very extensive.

In the course of its application to alternating-current problems the method of complex quantities has become closely related to circular trigonometry. The theory of circular and hyperbolic functions of a complex angle is of considerable importance in electrical theory ‡, and has been applied to numerous problems in this connexion ; thus it may be regarded as an extension of complex methods, and must be included in our bibliography of the subject.

In the appended bibliography, arranged in historical order, are given the most important relevant papers and

* In a letter addressed to Dr. E. W. Rice, Jr., discussing this writer's article on "Charles Proteus Steinmetz" in the 'General Electric Review,' xxvi. no. 12, pp. 796–799 (Dec. 1923). *Vide* Gen. El. Rev. xxvii. no. 2, pp. 132–133 (Feb. 1924).

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‡ The 'Tables of Complex Hyperbolic and Circular Functions,' by A. E. Kennelly, Harvard University Press, contains functions of $\rho/8$ to $\rho=3{\cdot}0$ by steps of $0{\cdot}1$ and to $\delta=45°$ to $90°$, together with functions of $x+iq$ to $x=10$, by steps of $0{\cdot}05$, and to $q=\infty$ (virtually) by steps of $0{\cdot}05$.

The same author's 'Chart Atlas of Complex Hyperbolic and Circular Functions,' Harvard University Press, contains curves of the above-mentioned calculations, to facilitate graphical computation.

works concerning both pure and applied theory. It must, however, be pointed out that the list is of necessity incomplete on account of the dissociated nature of the majority of these publications and the consequent difficulty of obtaining references thereto.

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LXXXII. *Fine Structure of Zinc Lines in the Visible and the Ultra-Violet Regions.*

By Wali Mohammad, *M.A. (Punjab), B.A. (Cantab.), Ph.D. (Göttingen), Professor of Physics, Lucknow University, and* Prem Nath Sharma, *M.Sc., Fellow, Lucknow University* *.

I. *Introduction.*

In earlier papers published in the Philosophical Magazine one of the authors has investigated the fine structure of arc lines in vacuum of cadmium and thallium [13] in the ultra-violet. The following is the extension of the same work to zinc.

II. *Previous Work.*

Michelson [1], Houston [2], Hamy [3], Janicki, [4,9] Gehrcke and von Baeyer [5], Lunelund [6,10], and Wali Mohammad [7] also

* Communicated by the Authors.

have worked on the same element, but they have confined their attention to the visible region only.

As will be seen later on, there is hardly any information available on the fine structure of zinc lines in the ultra-violet.

III. *Apparatus Used.*

The apparatus used was the same as was described in the previous papers. Briefly speaking, the light from a special vacuum arc lamp consisting of a Wehnelt oxy-cathode of platinum and a zinc anode, after having been rendered parallel by a quartz lens, was made to fall on a quartz Lummer-Gehrcke plate, after passing which the beam was accurately focussed by a quartz-fluorite achromat on the slit of the spectrograph. Two quartz Lummer-Gehrcke plates were available for use, and had the following dimensions :—

Small plate :—Length 13·0 cm.
Thickness 0·4597 cm.

Large plate :—Length 19·5 cm.
Thickness 0·365 cm.

The spectrograph used was the new 3-metre type with interchangeable optical system introduced recently by Hilger, and had a quartz prism and a quartz lens of 7·5 cm. aperture. This gave a very good dispersion in the regions investigated, and was a much more satisfactory arrangement than the Hilger E_3 spectrograph used in previous investigations. A Wollaston prism was placed in front of the Lummer plate in order to transmit the beam polarized with the electric vector parallel to the parallel surfaces of the Lummer plate. The fringes were photographed in the usual way in the spectrograph. The region from 8000 Å.U. to 2680 Å.U. was covered by three plates each 10 cm. long. Ilford Panchromatic plates for the visible region and Wellington Iso-and Anti-screen plates for the ultra-violet region were used. The time of exposure was generally about 5 minutes except in the case of faint lines, when the exposure had to be made longer. The oxy-cathode in the lamp was heated to incandescence by a current of about 15 amperes, and a voltage of 220 was applied between the cathode and the anode made of zinc. The thermionic current varied from ·3 to ·6 amp.

IV. *Calculation of Wave-lengths.*

Measurements were taken upon the fringe system from the upper face of the Lummer plate, and the fringes of the lowest

orders were measured and the mean of several observations taken. In this way the distance between two main-line fringes and between a main-line fringe and the satellites was determined.

The distance $\Delta\lambda$ max in Å.U. between the orders of the fringe system of spectrum lines was calculated from the constants of the Lummer plate with the help of von Baeyer's formula,

$$\Delta\lambda \text{ max.} = \frac{n\lambda^2}{n^2\lambda - 4t^2\mu\frac{d\mu}{d\lambda}},$$

where

μ = the refractive index of quartz with respect to the ray under investigation,

n = order of the fringe system given by

$$n\lambda = 2t\sqrt{\mu^2 - 1};$$

t = thickness of the Lummer plate;

λ = wave-length of the ray; and

$\frac{d\mu}{d\lambda}$ is the dispersion of quartz about λ.

The position of the satellites was determined in the following manner :—

If a is the distance between two successive main-line fringes, and b is the distance between the satellite and the lower-order fringe, then the wave-length of the satellite is

$$\frac{b}{a}\Delta\lambda \text{ max.}$$

If $b < a/2$, the satellite is positive and if $b > a/2$ the satellite is negative.

V. *Measurement on Zinc Lines.*

1. λ 6362·58.

Michelson [1], Houstoun [2] Janicki [4, 9], Lunelund [6, 10], Gehrcke and von Baeyer [5 8], and Wali Mohammad [7] all find this line to be sharp and simple. Nutting [11] obtained this line simple only when extremely sharp. He found it triple with diffuse components separated by about 0·80 t.m. from the central primary.

We find this line to be extremely sharp and simple.

2. λ 5182·17.

Janicki [4, 9] and Wali Mohammad [7] found this line to be sharp and simple.

We also find this line to be sharp and simple.

3. λ 4810·71.

Houstoun [2], Lunelund [6, 10], Gehrcke and von Baeyer [5, 8], Janicki [4, 9], and Wali Mohammad [7] find this line to be sharp and simple. Michelson [1] found that it possessed one satellite, and Nutting [11] found it to possess a trace of structure on reversal.

We find this line to be sharp and simple.

4. λ 4722·26

Gehrcke and von Baeyer [5, 8], Janicki [4, 9], and Wali Mohammad [7] find this line to be sharp and simple. Houstoun [2] found this line to be double. Nutting [11] found that, like λ 4810·71, this line also showed a trace of structure on reversal.

We find that this line possesses one satellite :—

+0·070
0·000.

5. λ 4680·38.

Gehrcke and von Baeyer [5, 8], Janicki [4, 9], and Wali Mohammad [7] find this line to be sharp and simple. Houstoun [2] found it to be double, and Nutting [11] noticed a trace of structure on reversal, as in the case of λ 4810·71 and λ 4722·96.

We find this line to be sharp and simple.

6. λ 4298·54.

This line is found to possess two satellites. One of these is strong while the other is very faint and diffuse, appearing only on certain plates. The following structure is obtained :—

+0·067 (very faint),
+0·042
0·000.

Sometimes the faint line appears at −0·046. Consequently the position of the faint satellite is doubtful.

7 & 8. λλ 3345·62 and 3345·13.

According to Kayser and Runge, there are two lines near each other and having the wave-lengths

3345·62,
3345·13.

We find that the line 3345·62 is simple, while the line 3345·13 possesses one satellite :

+0·033
0·000.

9 & 10. λλ 3303·03 and 3302·67.

Kayser and Runge have found two lines close to each other and of the wave-lengths

3303·03,
3302·67.

and of equal intensity.

We find that the line 3303·03 has one satellite :

0·000
−0·051.

The other line, 3302·67, is found to be simple.

11. λ 3282·42.

This line is sharp and simple.

12. λ 3075·99.

Wood [12] has found this line to be simple.

We find this line to possess one satellite which is quite sharp.

+0·033
0·000.

13. λ 3072·19.

This line appears simple.

14. λ 3035·93.

This line appears to have one satellite:

0·000
−0·056.

15. λ 3018·90.

This line appears to be sharp and simple.

16. λ 2800·90.

This line also appears sharp and simple.

17. λ 2770·94.

This line is found to have one satellite:

+0·042
0·000.

18. λ 2756·53.

This line is sharp and simple.

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LXXXIII. *Discharges in Neon.* *By* P. Johnson, *M.A.*, *Fellow of Magdalen College, Oxford* *.

1. FOLLOWING on researches which have been made at the Electrical Laboratory, Oxford, on the electric forces required to maintain high-frequency discharges in certain gases, I have undertaken the measurement of these forces in neon.

For these investigations it is necessary to have the gas very pure, and the precautions which were taken in order to obtain pure neon were the same as those used in other experiments which have been made in the laboratory and have already been published. Each specimen of the gas was tested for impurities by observing the spectrum of a high-frequency discharge in a wide tube containing the gas at high pressure. It has been found that this method of detecting impurities is much more sensitive than the ordinary method of examining the spectrum of a discharge in a narrow tube.

The discharge-tubes were cleaned by repeatedly admitting clean gas, heating with a blowpipe, and pumping the gas out again. After the process had been repeated several times, there were no impurities observed in the spectrum of the high-frequency discharge.

2. The arrangement first used for measuring the high-frequency currents in the tubes was exactly similar to one that was employed by Townsend and Nethercot † in their experiments on nitrogen. A continuous wave generator, maintained by two 30-watt valves, was used to induce

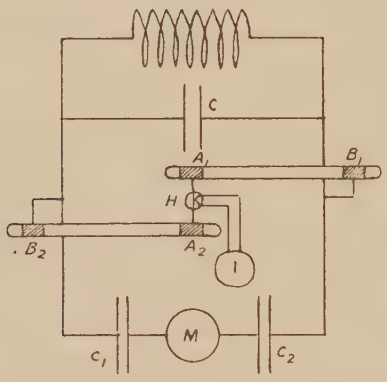
* Communicated by Prof. J. S. Townsend, F.R.S.

† J. S. Townsend and W. Nethercot, Phil. Mag. vii. p. 600 (March 1929).

oscillations in a secondary circuit, which is shown in fig. 1. It consisted of a symmetrical arrangement of two discharge-tubes fitted with external sleeve electrodes A_1, B_1, and A_2, B_2, the current, being measured by a thermocouple H situated between the two sleeves A_1, A_2. The currents in the tubes were adjusted by a rheostat in the filament circuit of the generator.

The distance x between the sleeves A_1 and B_1 was the same as the distance between A_2 and B_2, and the force was measured by observing the mean potential difference between the two sleeves B_1 and B_2 for a given current and different distances x between the sleeves. The potentials were measured by means of a thermoammeter M connected

Fig. 1.



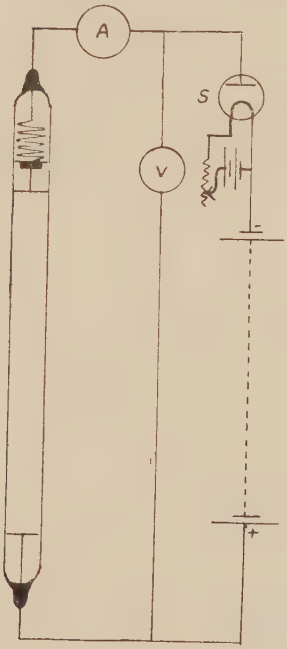
between two exactly similar variable condensers C_1 and C_2 in the secondary circuit, to which the sleeves were connected. The tubes were of pyrex of internal diameter 3·1 cm.

The results of these experiments showed that, as in the case of nitrogen, the mean force in the luminous column of a high-frequency discharge in neon was independent of the current in the tube from 1 to 10 milliamps, and of the wave-length of the oscillator from 30 to 120 metres.

3. In order to ascertain whether the force in the uniform positive column of a direct-current discharge is the same as the mean force in the luminous column of a high-frequency discharge, a pyrex tube of internal diameter 3·1 cm. was made up with internal aluminium electrodes, one of which

was fixed and the other movable along the axis of the tube. Electrical connexion was made to the movable electrode by a long strip of aluminium foil, and a piece of soft iron was fixed to the electrode so that it was possible to adjust the distance between the two electrodes by altering the position of the movable electrode with an electromagnet. A discharge was maintained by a battery of cells, and the current and the potential difference between the electrodes

Fig. 2.



were measured directly by means of a milliammeter A and a voltmeter V as shown in fig. 2.

The current in the tube was controlled by the two electrode valves S. The forces obtained were substantially the same as those found in the tubes with external electrodes with the same gas-pressures.

4. As this method of investigation, involving the use of three tubes, is rather complicated, it was considered desirable to make the measurements of the forces in the

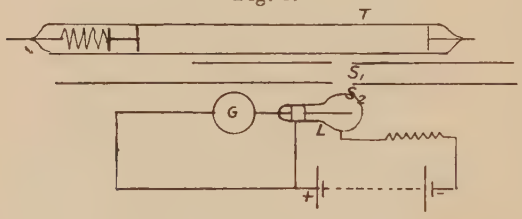
direct-current discharges and the high-frequency discharges with the same sample of the gas in the same tube. In order to do this, it became necessary to alter the method of measuring the currents in the high-frequency discharges so as to be able to use one discharge-tube only.

As there are some difficulties in obtaining accurate results by the method used by Townsend and Nethercot to measure currents in a single tube, another method was adopted which was found to be very convenient.

The current in the gas was indicated by a photoelectric cell on which the light from a small section of the discharge was allowed to fall. Since practically all the light from a neon discharge is confined to the red end of the spectrum, the photoelectric effect was small, and a sensitive galvanometer was required to indicate the photoelectric currents.

The discharge-tube T and the photoelectric cell L are arranged as shown in fig. 3, so that the light from the tube

Fig. 3.



which passes through the slits S_1 and S_2 falls on the cell. The internal electrodes of the tube could be connected as desired to the same sources of steady or high-frequency potentials as those used in the first experiments, and the methods of measuring these potentials were identical with those already described. It was also possible to maintain a high-frequency discharge in the tube T with external sleeves as electrodes, and in a preliminary set of experiments it was ascertained that the forces in the high-frequency discharges were the same whether the internal electrodes or the external sleeves were used.

The cell was calibrated by observing the deflexions of the galvanometer G when direct-current discharges were maintained in the tube T. The current in the galvanometer depended on the intensity I of the current in the tube and on the pressure p of the gas. It might be assumed that similar photoelectric currents would be obtained when

high-frequency discharges of similar intensities are maintained in the tube, but it is not necessary to make this assumption in order to measure the force in the luminous column of the high-frequency discharge. The mean potential difference E between the electrodes depends on the intensity I of the current through the gas; and the force X in the uniform luminous column is obtained by finding the potentials E_1 and E_2 corresponding to two different distances x_1 and x_2 between the electrodes. The force X in the luminous column is $(E_1—E_2)/(x_1—x_2)$, and experiments have shown that the force is independent of the current. Any current may therefore be used to determine X, provided the intensity I is the same when E_1 and E_2 are being measured. Since there are changes in the photoelectric current accompanying changes in the current in the discharge-tube, the latter may be maintained constant by adjusting the intensity of the oscillations so that the same photoelectric current is obtained when E_1 and E_2 are being measured for two distances x_1 and x_2 between the electrodes.

5. Table I. gives the values in volts per cm. of the forces x_1 in the uniform positive column of the direct-current discharge, and the mean forces X_2 in the uniform column of the high-frequency discharge of wave-length 35 metres, for different values of the pressures (p), which are given in millimetres of mercury. (The ratio of the mean force X_2 to the amplitude of the high-frequency force is $2/\pi$.) The currents in the tube were from 3 to 20 milliamps.

It is clear from the second and third columns that the forces in the two cases are practically identical until low pressures are reached. Below 2 mm. pressure the force in the high-frequency discharge remains steady at about 3 volts per cm. until striations set in at about ·5 mm. pressure.

The force in the positive column of the direct-current discharge decreases as the pressure decreases below 5 mm. and the column itself begins to lose its uniform appearance. At ·5 mm. pressure the force becomes very small, and it is difficult to measure since the total potential between the electrodes becomes very large at the lower pressures.

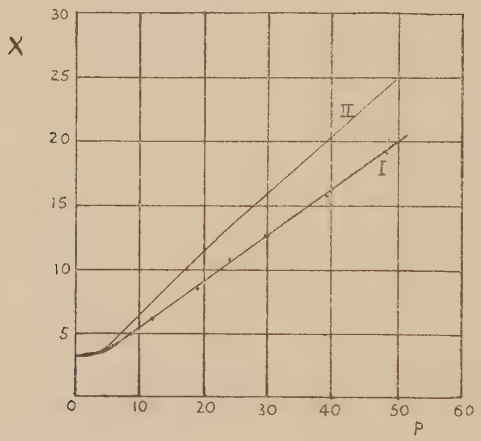
The variation of the mean force X in the high-frequency discharge with the pressure p is shown by curve I. (fig. 4), X being the ordinate in volts per cm., and p the abscissa in millimetres of mercury. The straight part of the curve

from $p=10$ to $p=50$ does not pass through the origin, and it is clear that the ratio X/p increases progressively as the pressure falls.

6. Values of the force in the luminous column of a high-frequency discharge in neon at different pressures have been determined by Hayman *, and for comparison his results are given in curve II. (fig. 4).

In his investigations the forces were not measured for different currents through the gas, but for the smallest current which gave a uniform glow in the tube, and he had no direct means of measuring this current. The accuracy of his measurement of the force depends on the fact that

Fig. 4.



this current is approximately the same for different distances between the electrodes, but it is clearly better to have a direct means of measuring the current and to find the values of X for different currents.

His values are greater than mine except at the lowest pressures, and this to a small extent is accounted for by the fact that the diameter of his tube was 2·9 cm. while that which I used was 3·1 cm. in diameter, but there are possible sources of error which may account for the difference between his results and mine.

The principal cause of the difference between the two sets of results is probably due to the fact that it is not

* R. L. Hayman, Phil. Mag. vii. p. 586 (March 1929).

possible to obtain neon which is absolutely free of helium, and the presence of the latter would have the effect of increasing the force.

The mean energy of agitation E_1 of electrons in gases has been determined in terms of X/p from the experiments on the diffusion of electrons in gases that have been made in the laboratory. The curves* giving the values of E_1 in helium and neon corresponding to the range of X/p given in Table I. show that for these values of X/p the mean energy of agitation is controlled to a great extent by the small losses of energy of the electrons in their collisions with atoms of the gas. These losses are greater in the collisions with atoms of helium than in the collisions with atoms of neon, so that a small amount of helium in the neon would have a comparatively large effect in reducing the energy of agitation. This effect is emphasized by the fact that the mean free paths between the collisions in which these small losses occur are smaller in helium than in neon. Taking these effects into consideration, it is to be expected that for a given value of the ratio X/p the energy of agitation of electrons in pure neon would be reduced by about 6 or 7 per cent. by introducing 1 per cent. of helium. Thus, in order that the supply of electrons by the process of ionization by collision may be sufficient to maintain a current in a discharge through neon containing 1 per cent. of helium, it would be necessary to have the electric force 6 or 7 per cent. greater than the force in pure neon.

7. In the process of purification the gas was stored at high pressure in a reservoir containing charcoal cooled with liquid air. Neon at high pressure is absorbed by the charcoal to a much greater extent than helium, but is given off again as the pressure is reduced. Small quantities of gas could be admitted to the discharge-tube from the reservoir, and it is clear that if there were any helium in the gas originally, the first samples so removed would contain a higher percentage of helium than the later samples. It was found that the force when measured in a discharge in one of the first samples of gas was invariably greater than that measured in a later sample.

The values given in Table I. and represented by curve I. (fig. 4) were obtained from what were considered to be the

* J. S. Townsend, Phil. Mag. ix. p. 1145 (June 1930).

purest specimens of the gas; but forces as much as 15 per cent. greater than these were sometimes obtained with the first specimens of the gas taken from the reservoir.

The average partial pressure of the helium in the various samples was about 1 per cent. of that of the neon. In the purest samples the partial pressure of the helium was considerably less.

8. In general the impurities in helium and neon are more easily detected by the radiation from discharges in wide tubes containing the gases at high pressures, but there is an exception to this rule in the case of neon when the impurity

Table I.

Forces X in luminous columns of discharges in neon at pressure p in a tube 3·1 cm. internal diameter.

p.	X_1.	X_2.	X/p.
48·0	19·5	19·0	·40
39·5	16·5	15·5	·40
30·0	12·5	12·7	·42
19·5	9·0	8·8	·46
13·0	7·0	6·8	·53
8·8	5·1	5·05	·58
6·4	4·3	4·1	·66
4·1	2·8	3·6	—

consists of a small quantity of helium. In this case the helium is more easily detected when the gas-pressure is low. This may be attributed to the fact that the energy of agitation of the electrons in the gas, which increases with the ratio X/p, is greater at low pressures than at high pressures. (The figures in Table I. show that the ratio X/p is greater for small pressures than for large pressures.)

I have examined mixtures of helium and neon spectroscopically, using a high-frequency discharge of wave-length 35 metres. In a mixture where the partial pressure of helium was three times that of the neon, it was not possible to detect the lines due to helium by a direct-vision spectroscope when the total pressure in the discharge-tubes was greater than 5 mm. In fact, the radiation from the discharges at the higher pressures appeared to be the same as that from a discharge in pure neon. When the pressure

was reduced to 3 mm. the principal blue line of the helium spectrum could be seen.

In another mixture of about equal parts of helium and neon the helium lines were faintly discernible at a pressure of 1·5 mm. It follows from the results of these experiments that a small trace of helium in neon could not be detected except at a very low pressure, but here a further difficulty would arise, namely that the intensity of the light emitted from the discharge would be so small that no impurities could be identified.

9. When calibrating the photoelectric cell by means of the radiation from the positive column of a direct-current discharge in which a known current was flowing, the photoelectric current through the cell was found to be proportional to the current in the tube. This means that

TABLE II.

Photoelectric current d in arbitrary units for a discharge current I in milliamps. in neon at 10 mm. pressure.

I.	d.	I/d.
14·0	4·5	3·1
10·7	3·5	3·05
7·6	2·5	3·05

the intensity of light emitted is proportional to the current, as was found by Gill * for the ultra-violet light which is emitted by a discharge in air. This result is in accordance with the theory of the positive column, which shows that the effect of recombination is negligible †. Since the electric force is independent of the current, the energy of agitation of the electrons is also independent of the current, and all effects due to collisions of electrons with molecules of the gas are proportional to the current.

The calibration of the photoelectric cell by means of the direct-current discharge in neon at 10 mm. pressure is shown in Table II., where I is the current in the discharge-tube in milliamps., and d the deflexion of the galvanometer

* E. W. B. Gill, Phil. Mag. p. 412 (Sept. 1911).

† J. S. Townsend, 'Electricity in Gases,' pp. 441 and 442; *Comptes Rendus*, p. 55 (Jan. 1928).

measuring the photoelectric current in centimetres; the third column gives the ratio I/d, which is seen to be constant and equal to 3·1 when the gas-pressure is constant at 10 mm. This ratio fell to 2·5 when the pressure in the tube was 20 mm., and had about that value at all higher pressures, the behaviour of the cell varying a little from day to day. As the pressure was decreased from 10 mm. the ratio increased and was about 7 at 2 mm. pressure. It appears from this that the intensity of light for a given current emitted from a direct-current discharge is practically constant at high pressures, but at lower pressures the intensity decreases as the pressure decreases. The colour also changes, and the proportion of red light to yellow light diminishes as the pressure is reduced. Further experiments on the intensity of the light in different parts of the spectrum are in progress.

10. A selenium cell has since been tried in place of the photoelectric cell, and was found to be far more sensitive to red light, giving a current of the order of $\frac{1}{10}$ milliamp. when 10 milliamps. were flowing in the discharge-tube. It has the minor disadvantages that a considerable current flows through the selenium when no light is falling on it, and that the increase of the current in the cell is not exactly proportional to the current in the discharge-tube.

11. In a tube 3·1 cm. in diameter the luminous column of the high-frequency discharge in neon begins to break up into striations at a pressure of ·4 mm. The striations take the form of a series of regions of high and low luminosity along the length of the luminous column. The number of these regions depends on the distance between the electrodes, and the relative brightness of the different sections of the luminous column increases as the pressure is diminished.

At pressures in the neighbourhood of 5 mm., using external sleeve electrodes, spiral striations of the form described by MacCallum * were obtained. They appeared after the gas had been rendered impure by strongly heating the discharge-tube and then allowing the high-frequency discharge to carry away most of the impurity. They were never obtained in pure neon.

* S. P. MacCullum and W. T. Perry, 'Nature,' cxxiv. p. 984, (Dec. 28, 1929).

12. It was observed when the current in the high-frequency discharge in neon was measured by means of the thermocouple, that at low pressures there was always a certain minimum value of the current that would flow. The value of this current increased as the pressure decreased. Thus at a pressure of ·4 mm. the current could not be reduced below 2·5 milliamps. without the discharge ceasing altogether; at ·15 mm. pressure the minimum current was 4 milliamp. The wave-length of the oscillations in these experiments was 35 metres.

My best thanks are due to Professor Townsend for his advice and help throughout this research.

LXXXIV. *On the Principle of the Inaccessibility of the Absolute Zero.* *By* Heinrich Mache, *Ph.D., Professor of Physics at the Technical High School, Vienna* *.

Prof. N. A. Kolossowsky has recently observed † that Clausius gave expression to this principle as early as 1862, and objects that Nernst and all other authors who have been concerned with this problem at a later date have not given him sufficient credit. He writes: "It is strange enough that nobody has up to date mentioned the circumstance, so much the more that, if one occupies oneself in researches in the domain of these questions, a deep study of Clausius' classical works is absolutely indispensable." As one of the authors referred to by Prof. Kolossowsky perhaps I may be allowed to point out that what he says is not justified, since to-day one recognizes a new postulate in the said principle, and no longer a simple deduction from the second law as it appeared to Clausius. Actually the principle can only be deduced from the second law when it is assumed that the specific heat remains finite at the absolute zero, or, if it vanishes, it does so at a lower order than T. This is shown ‡ by the following consideration:

Let us consider a substance which passes through a reversed Carnot cycle, *i. e.*, which behaves as a refrigerating machine, in that it extracts the amount of heat q from the

* Communicated by Dr. Robert W. Lawson.

† Phil. Mag. (7) ix. p. 208 (1930).

‡ *Cf.* also *Ber. Wien. Akad. Wiss.* (II. *a*) cxxxvi. p. 75 (1927).

heat container at lower temperature by the expenditure of work $q'-q$. Then $q/q'-q=T/T'-T$ is a measure of the efficiency for each cycle. As heat container at the lower temperature we consider the body which is to be cooled, so that T is variable; on the other hand, the upper isotherm we keep unchanged at the temperature T'. The efficiency per cycle decreases with T and converges towards zero as the absolute zero is approached. Nevertheless, the absolute zero might be attainable with finite expenditure of work, provided that the specific heat of the body being cooled decreased also towards the value zero.

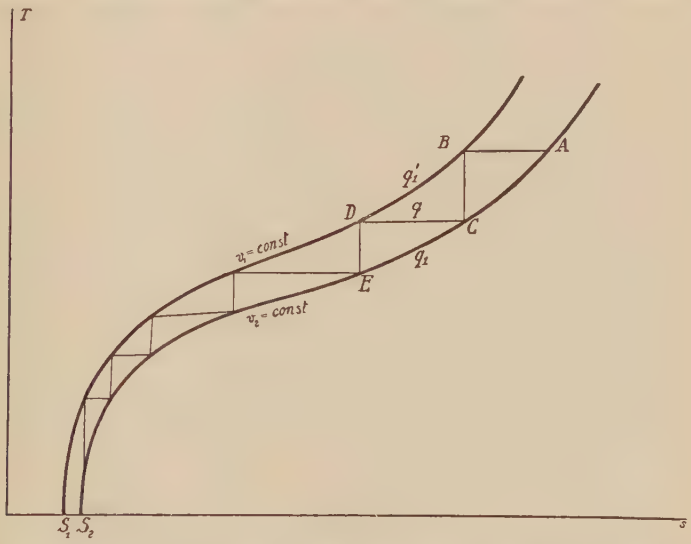
In order that the removal of heat in each cycle may occur at a definite temperature, and the cycle may remain perfectly reversible also with respect to the transfer of heat, let us make the assumption that the cooling body during the removal of heat is simultaneously subjected to an isothermal compression, whereby the quantity of heat q is given up to the working substance in the Carnot process, and that it is then expanded again adiabatically to the original volume. In this way its temperature falls in stages from cycle to cycle, but does not change so long as it is in contact with the working substance. The isothermal compression and adiabatic expansion occur in this way at each cycle between the same limits of volume, let us say from v_1 to v_2. The heat of compression q, developed and taken up by the working substance, will be different at each cycle, and in accordance with the relation $q=q' \cdot T/T'$ will decrease towards zero with decrease in T.

We can represent the changes which the cooling body experiences in this way in a Gibbs diagram, after Grumbach *. If the curves represent lines of constant volume, between which the body is alternately isothermally compressed (AB, CD, etc.) and adiabatically expanded (BC, DE, etc.), a series of steps of decreasing height is obtained, provided that at higher temperatures c_v remains approximately constant, and the lines of constant volume are represented fairly well by logarithmic curves. If however c_v is small at low temperatures and finally zero, so that the curves meet the s-axis normally, the height of the steps will again increase, and it no longer appears

* *L'Enseignement scientifique*, i. p. 139 (1928).

impossible that, despite the simultaneous decrease in efficiency of the Carnot process, the absolute zero can be reached *.

If we denote by q_1 that quantity of heat which must be removed from the body to bring it from C to E on the curve v_2=const., and by c_v the mean value of its specific heat on this path, then q_1 is smaller than q by the area DCE, and the cooling resulting from the adiabatic expansion of the body $q_1 c_v$ is smaller than q/c_v. If q/c_v disappears as the absolute zero is approached, then the cooling obtained by adiabatic expansion to the initial volume after each cycle of the Carnot process



will decrease still more towards zero. Since, however, the diminution of $q=q'$. T/T′ is of the same order as T, the quotient q/c_v will disappear if c_v is *finite* or *decreases more slowly than* T *towards zero* †. Thus an infinite expen-

* According to Nernst's theorem S_1 and S_2 will coincide, but from the second law alone we must consider s as a function of v.

† It is obvious that q' must remain finite. Otherwise two adiabatics starting from two points on an isotherm as far apart as $v_2 - v_1$ would meet at the temperature T′ and trace a cycle of finite area without consumption of heat.

The same can be shown in the T—s diagram. As long as S_1 and S_2 do not coincide the area over each of the isothermal lines (*e. g.*, CD) up to the temperature T′, representing q', will be finite. It is only from Nernst's theorem that we know that S_1 and S_2 coincide, and the area can become infinitely small.

diture of work is required in order to cool the body to the absolute zero, and its unattainability is a consequence of the second law alone, or, more exactly, a consequence of the assumption that the second law still holds, however closely we approach the absolute zero.

The proof is convincing, since the reversible Carnot cycle represents the most efficient refrigerating machine. One might perhaps raise the criticism that the expansion occurs always from the volume v_1 to the volume v_2, and maintain that at the lowest temperatures use should be made of larger expansions. However, since nothing more has been said regarding the difference $v_2 - v_1$ than that it is neither infinitely small nor infinitely large, nothing prevents our assuming the largest possible expansion during the whole cooling process.

If we admit that the specific heats vanish at a higher order than T the case is quite different, and the absolute zero appears attainable. To show this we make the assumption that now the cooling body expands at first adiabatically along BC, comes in contact with the working substance of the Carnot cycle, and is then compressed at constant temperature (CD) till the original volume is again reached. If q_1' denotes the quantity of heat which we must remove from the body on the curve $v_1 = \text{const.}$ from B to D and c_v' the mean value of its specific heat on this path, then q_1' is greater than q by the area BCD, and the cooling produced by the adiabatic expansion $q_1'/c_v' > q/c_v'$. As now q/c_v' rises to infinitely large values, q_1'/c_v' will do so *a fortiori*.

Clausius did not hesitate to express the principle that the absolute zero was unreachable, since in his time one could not foresee that specific heats decrease towards zero as the absolute zero is approached. In the paper quoted by Kolossowsky, which Clausius did not retain, moreover, in the second and third editions of his 'Mechanische Wärmetheorie,' a proof is given which is characterized by the introduction of the concept of "disgregation" (z). By this is understood the quotient of the work performed against external and internal forces for a change of state, and the temperature. On a reversible adiabatic the increase in disgregation on expansion from the temperature T_0 to T is given by the expression

$$z - z_0 = \int_{T_0}^{T} \frac{k\,dT}{T},$$

where k is what Clausius calls the "true heat capacity." It denotes by how much the kinetic energy per unit-mass of the substance under consideration increases per degree, and should be independent of the temperature and the state of aggregation, which, as we know to-day, is only the case when the number of degrees of freedom remains unchanged and the principle of equipartition remains valid. If from the fact that when $T=0$ the change of disgregation becomes infinitely large it is concluded that the absolute zero is unattainable, then it must be assumed that the true heat capacity k either remains finite or disappears at a slower rate than T. But k is only a part of c_v or c_p, and if these quantities disappear then k must also disappear.

What has been so far considered may be summarised as follows :—If it is assumed that the second law still holds at the absolute zero, then the absolute zero is unattainable *only* for the case when the specific heat remains finite or disappears at a slower rate than T. If it disappears according to an expression of higher order than the first, then the zero appears to be attainable.

The original conception of Nernst's heat theorem requires that at the absolute zero the specific heat remains unaltered for every change. Planck goes somewhat further in his hypothesis, which requires that the heat capacity disappears at the absolute zero, as had been confirmed by Nernst. Not much has been said here of the order in which the specific heat disappears. A finite zero-point value for the entropy only presupposes that when $T=0$, $\int_0^T \frac{c\,dT}{T}$ converges towards zero. If we take $c=aT^n$, then $\int_0 \frac{c\,dT}{T} = \frac{a}{n}T^n$, and this expression disappears for all positive values of n, and also for values of n less than one.

It is, however, to be assumed on the basis of experimental work and of Debye's theory of specific heats that c disappears more rapidly than T. If we bring this into consideration, we can with Nernst extend the heat theorem in the following words :—"*In the neighbourhood of the absolute zero every process takes place without change in the entropy*" *. Thus for $T=0$, and for every change taking

* W. Nernst, 'Die theoret. u. exper. Grundlagen des neuen Wärmesatzes,' I. Aufl. p. 71 (1918).

place there or starting off from there, we have $ds=0$, where ds represents the *total differential* of the entropy. In actual fact it follows conversely from $ds=dq/T=cdT/T=0$, and $ds/dT=c/T=0$, for $T=0$, that c disappears more rapidly than T. This means that not only every change of pressure, density, or chemical union occurring on the zero isotherm, but also an infinitely small rise of temperature taking place on it, leaves the value of the entropy unchanged. Not only is the zero isotherm itself an adiabatic, but also every change of state starting from any point on the zero isotherm proceeds at first adiabatically. Herein, however, is revealed the contradiction in which the behaviour of this region lying on the zero isotherm stands to the second law ; adiabatic curves will actually intersect there in a finite range. The principle of Caratheodory * can also be utilized, according to which, in the region of validity of the second law, there must exist, in arbitrary proximity to every state, neighbouring states which cannot be reached from the first state on an adiabatic path. On the other hand, from the zero-point isotherm all neighbouring states can be reached solely by an adiabatic path. One can say, therefore, that in the immediate neighbourhood of the absolute zero the second law loses its validity for every body. *Only as a result of this does it appear possible that, in spite of the disappearance of specific heats in an order higher than the first, the zero nevertheless remains unreachable.* The proof given above, according to which the zero was attainable in this case, assumed the validity of the second law also at the absolute zero, and now appears untenable.

If the unattainability of the zero-point is also postulated, with Nernst, for the case in which the specific heats disappear with T in higher order than the first, then this postulate demands conversely the invalidity of the second law, and with it the constancy of the entropy in the region lying near the zero isotherm. It is therefore a necessary consequence of this postulate that for $T=0$, $ds=0$.

From the previous considerations one is led to the conviction that it would have been inexpedient to describe the principle of the unattainability of the absolute zero

* *Cf.* M. Born, *Phys. Zeits.* xxii. p. 282 (1921).

as a special principle as long as it appeared wholly contained in the second law, and in the "Clausius-Carnot principle," from which this second law was developed. It is the experiments of Nernst on specific heats at the lowest attainable temperatures, and the related theoretical investigations of Einstein and of Debye, which have raised the principle of the inaccessibility of the absolute zero to an independent postulate, with a wider physical meaning than hitherto. It would therefore be unjust not to name this new knowledge after Nernst, but to attach to it the name of Clausius, the more so as to-day it is generally accepted that ultimately all the essentials of the new thermodynamics depend on the fundamentals which were laid down by Clausius.

LXXXV. *The Effect of Systematic Surface Treatment on the Photoelectric Emission from Metals.* *By* R. F. Hanstock, *B.Sc., University College, Nottingham* *.

THERE is evidence to show that †, when an annealed metal surface is polished, a change takes place in the physical nature of the surface. This research is undertaken to examine the photoelectric emission from metal surfaces in a vacuum at intervals during the process of polishing, the surface being initially in an annealed state.

The following is a description of the apparatus used for this purpose :—Treatment of the metal surface is done in a vacuum tube AA (fig. 1) of "Vitreosil" (fused quartz). This tube is found to provide good insulation for the metal parts and can easily be heated to the temperature required to anneal the metal under examination. The end of the tube is closed by a transparent quartz plate attached by sealing-wax. During the heating of the tube the wax is kept cool by a current of cold air from a fan-motor. The metal to be tested is shown at M. This is a flexible strip, 5 cm. long, 5 mm. broad, and 0·07 mm. thick. The strip is fixed to a brass ring CC, held in a plane perpendicular to

* Communicated by Prof. P. E. Shaw, M.A., D.Sc.

† Beilby, 'Aggregation and Flow of Solids.' Shaw, Proc. Roy. Soc. A, xciv. p. 16 (1917); Proc. Phys. Soc. xxxix. p. 449 (1927). Shaw and Jex, Proc. Roy. Soc. A, cxi. p. 349 (1926).

the axis of the tube by means of the framework shown. M is connected through the lead-quartz seal B to the quadrant electrometer. Fig. 1 *a* shows a front view of the ring C. The anode of the photoelectric cell is the plate DD, having a slit S, 9 mm. long and 1 mm. wide, through which U.V. radiation falls on the specimen M. The distance between D and C is 4 cm., and the potential applied to D is +60 volts. To obtain good insulation between the high potential plate and the electrometer system an earthed metal ring, F, is placed round the tube near to C. Wires, G of copper, H of eureka, are connected by F, and pass through the insulating tubes K and L to the outside. This arrangement forms one junction of a thermo-couple, the

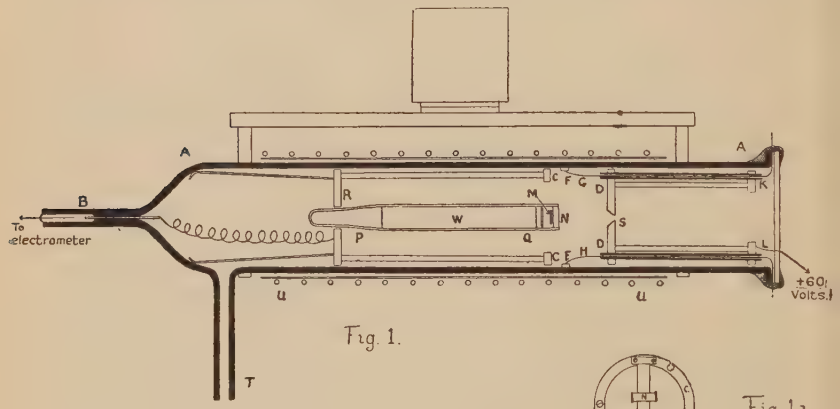


Fig. 1.

Fig. 1a.

other junction being kept in a bath of water outside the tube, and hence provides a means of measuring the temperature near the specimen. The thermoelectric circuit is earthed at one point.

To polish the specimen the metal cylinder N is used. This is held fixed at one end of the bar PQ, the other end of which fits loosely into a hole in the plate R. This bar is weighted by a mass of brass, W, weighing 200 gms. When the tube is held vertically, with the quartz window upwards, the weight is supported by the cylinder resting on the strip. If the tube is now rocked from side to side in the vertical plane containing the strip, N slides over M and polishes the latter. The strip, being very thin and

flexible, adjusts itself to maximum contact with the cylinder. The tube is mounted so that it can easily be oscillated to produce the polishing. In presenting the results obtained, a slide of the polishing cylinder from one end of the strip to the other is called 1 rub. The side tube T is connected to a vacuum pump, drying tubes, and McLeod gauge by means of a short length of pressure tubing, this being necessary to allow for the movement of the apparatus. The apparatus is heated by the coil U, the leads to which are earthed during electrometer readings. The whole apparatus and lead to the electrometer are electrostatically shielded.

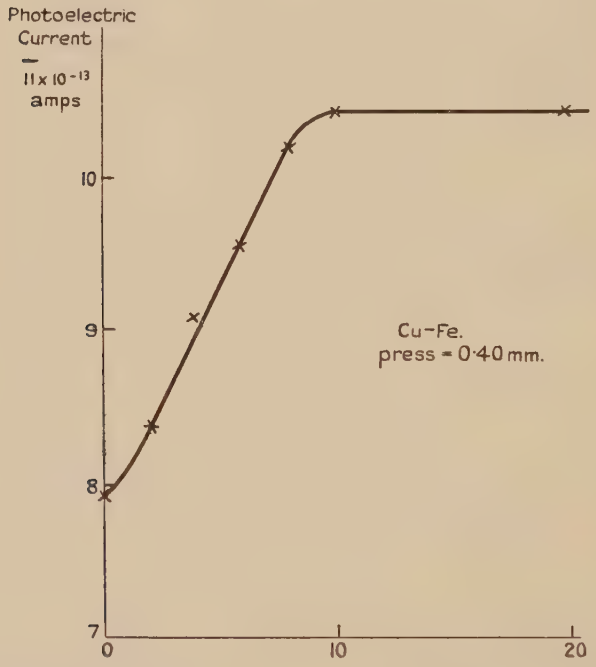
When exposing the metal to the U.V. radiation the tube is held horizontally, so that M falls to one end of the strip, which is thus left freely exposed to the light. As the source of U.V. radiation a parallel arc is used, the electrodes being made of an alloy of Ni and Cr silicates. This produces radiation of wave-length down to 1900 Å, and is an intense source. The arc is housed in an earthed metal case with a shutter, and is at a distance of 50 cm. from the specimen.

The following is a description of the preparation of the specimens and the method of taking readings. The specimen is cleaned with a fine glass-paper, wiped with clean filter-paper, and then washed in benzene. The cylinder is kept highly polished by means of rouge, and is washed in benzene before fixing in the tube. When sealed up the tube is heated to 300° C. and then allowed to cool slowly, remaining above the temperature 200° C. for approximately one hour, the tube being evacuated during heating. Further prolongation of the heating is not found to influence the results. After the tube has cooled down the photoelectric readings are taken. These consist in a measurement of the P.E. current when the surface is annealed, followed by measurement of the current as the surface is rubbed by increasing amounts. After rubbing, the apparatus is allowed to stand for a short time before taking readings, so that any frictional charge developed (indicated by the electrometer) may leak away. The method employed in measuring the current is to observe the time for the spot of light from the electrometer to drift through a 100 mm. on the scale, after the rate of drift has become constant. These measurements thus give a curve showing how the photoelectric current changes on

rubbing the surface. Examination of the surface after rubbing shows, for the material used, that it is polished and not simply scratched.

The materials used for the strips are Cu, Ag, Au, and Pt, and for the polishing cylinder steel and Ni. These are chosen because both steel and Ni polish the other metals easily without scratching, and are not likely to form alloys with them. It is not possible to use a polishing cylinder

Fig. 2.



made of the same metal as the strip, since it is found that a tearing and scratching of the surface results.

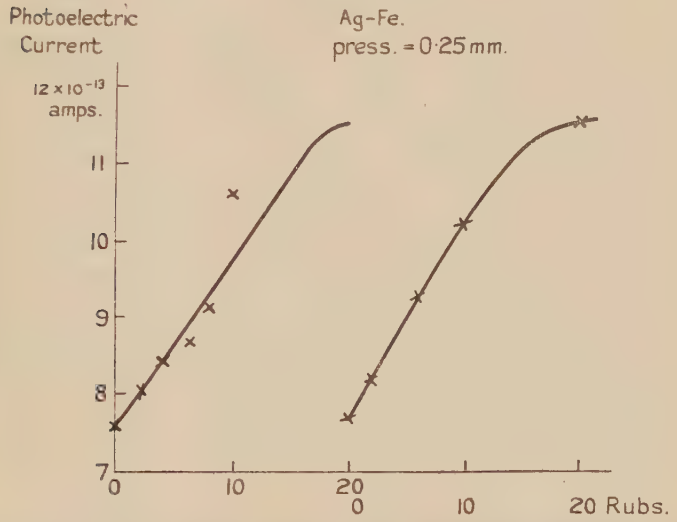
Results.

Fig. 2 is the curve for Cu rubbed with steel, the readings being taken at a pressure of 0·40 mm. All the metals tested give curves of similar shape. There is an increase of the photoelectric current as the metal is changed from an annealed to a polished state. The relation between photoelectric current and number of rubs is almost linear

for the first eight rubs, and then quickly reaches a stationary value.

In fig. 3 are shown curves for Ag rubbed with steel. The first curve is obtained in the usual way. The specimen is then heated again to the annealing temperature, and, on cooling, the readings are repeated, producing the second curve. It is found that the photoelectric current can always be increased by rubbing, decreased by heating, and then increased by rubbing, and so on for any number of times. As a result of experiments using Ag rubbed by

Fig. 3.



steel, it is found that unless the specimen is heated to more than 150° C. there is no change to the annealed state from the polished state as indicated by the photoelectric current.

There seems to be no definite relation between the pressure in the tube and the ratio of the current for the polished to the current for the annealed state. Calling the current from the surface in the polished state i_m, and the current from the annealed surface i_0, it is found that 23 curves taken for Ag, Pt, Cu, Au, rubbed with steel, and Ag rubbed with Ni, give values of i_m/i_0 between 1·85 and 1·15, but of these 23 curves 20 give values of i_m/i_0 between 1·51 and 1·15. The range of pressure used is from

0·001 mm. to 1 mm. The following are the mean values of i_m/i_0 for the various metal conbinations :—

Ag/Steel.....	1·41
Au/Steel.....	1·45
Pt/Steel.....	1·33
Cu/Steel.....	1·23
Ag/Ni.....	1·38

The experiment is also repeated for a more limited range of the exciting radiation. For this purpose a filter consisting of a solution of potassium chromate in water is used. The solution is contained in a Baly tube with quartz ends, so that the thickness of the absorbing solution can be varied. The sensitivity of the electrometer is increased to about ten times its previous value, and the thickness of the solution adjusted to the greatest value consistent with a measurable photoelectric current. Photographs of the arc spectrum taken through the solution show transmission between 2260 and 2330 Å., the limiting lines being very faint. Transmission begins again above 3000 Å., but since the long-wave limits for the metals used are probably not greater than 3000 Å., this should have little effect. With this limited range of wave-length the curve shown in fig. 4 is obtained for Ag rubbed with steel, the pressure being 0·15 mm.

The main characteristics of the curves may be summarized as follows :—

1. The photoelectric curent is increased when the metal surface is polished.

2. The increase of the photoelectric current with rubbing is approximately a linear function of the number of rubs until just before the maximum value is reached.

3. The ratio i_m/i_0 shows no regular dependence on the pressure of the gas in the tube.

4. The effect appears for radiation of a restricted range of wave-length as well as for the full spectrum.

5. By heating to above 150° C. the metal can be reduced to its less active state.

The following are some of the possible causes of the effect observed requiring consideration:—

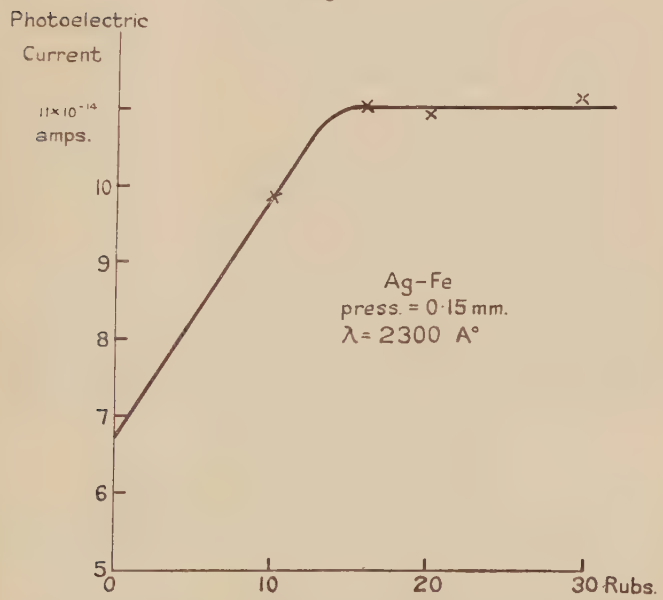
(*a*) A change in the adsorbed gas layer at the metal surface, brought about by the rubbing.

(*b*) Transfer of metal from the polishing cylinder to the specimen.

(*c*) The formation of oxide films by heating, and removal or transfer on rubbing.

(*d*) The formation of a layer of a modified form of the annealed metal, corresponding to Beilby's "vitreous" layer, produced by polishing.

Fig. 4.



While it is impossible to eliminate definitely any of the above possibilities, the results obtained make some more probable than others.

Since there is no definite dependence of i_m/i_0 on the pressure throughout the range used (0·001 to 1 mm.), and as also the increased activity produced by rubbing does not disappear even after several days, the effect does not seem to be due chiefly to a removal of part of the adsorbed layer. It is possible, however, that a change in the form of the metal surface itself, either geometrical or physical,

might alter its adsorbed layer, and thus increase the emission. If the change were due solely to a smoothing out of surface irregularities the reproducibility obtainable by successive heating and polishing would not be expected.

The possibility of transfer of metal from the polishing cylinder to the specimen is always present, but it should be noted that it would be a transfer from a harder metal to a softer one in all the experiments undertaken. Even if transfer takes place the reproducibility obtained by successive heating and polishing requires more than this to explain it, since heating to a temperature of 150° C. would not remove the transferred metal.

Formation of thin oxide films on the surfaces during heating is very probable, although no visible signs of oxidation are noticed. Also the values of i_{∞}/i_0 are nearly the same for the easily oxidizing metals Cu and Ag as for Pt and Au.

The remaining possible cause is worthy of consideration, since it will explain the results obtained, and since there is other experimental evidence indicating that a modified layer exists on the surface of a polished metal. Beilby has shown that such a layer can be produced to a depth of 50 to 500 μ on the surface of a metal by polishing. By heating to about one-third of the absolute melting-point of the metal this layer returns to its original state. The minimum temperature required to change the metal from the more to the less active photoelectric state is shown to be greater than 150° C., which is approximately one-third of the melting-points for the metals considered.

There are other physical properties of metals considered by Beilby which show a sudden change at this "annealing" temperature, namely, the electrical conductivity, thermal E.M.F., and elasticity. It has also been shown * that the tribo-electric charge developed when a metal is rubbed on an insulator depends on the state of the metal. Hence it is probable that the change observed for the photoelectric effect arises from the same cause as that producing change in these other physical properties.

In conclusion, I wish to thank Prof. P. E. Shaw, M.A., D.Sc., of the Physics Department, University College, Nottingham, for the facilities provided for this research and for advice in experimental details.

* Shaw, Proc. Roy. Soc. A, xciv. p. 16 (1917).

LXXXVI. *On the Effect of Constrictions in Organ Pipes.* By Eric J. Irons, *Ph.D.**

1. *Introductory.*

THE work described in this paper is a sequel to two previous investigations. The first of these[1] was designed to determine the effect on the distribution of nodes and antinodes of introducing diaphragm constrictions into a Kundt's sound tube, and was, in consequence, concerned with the adjustment of "tube lengths" for resonance to a note of fixed frequency. The second investigation[2] was on the effect of side-holes in wind instruments, and the present problem of determining the frequency variation of an organ pipe of fixed length when diaphragm constrictions are placed at various points along its axis is in theory similar to it. The theory of the problem has already been treated in part[3], and on it the present work was founded, originally with a view to designing an organ pipe capable of emitting a lower note than an ordinary stopped pipe of the same length. During the progress of the experiments Dr. E. G. Richardson drew my attention to a description of a patent in *Musique et Instruments*, and on investigation I found that the idea of realizing a bass organ pipe on this principle was the subject of French patent No. 621,719 (1927) granted to M. de Mons. I have accordingly restricted this report to the more complete theory proposed in the next paragraph, and experimental data to verify it.

2. *Theoretical.*

In a former paper[3] simple proofs were given of Webster's formulæ[4].

$$\text{(A)} \qquad Z_1 = \frac{\beta}{\sigma} \cdot \frac{Z_2 \cos kl - (\beta/\sigma)\sin kl}{Z_2 \sin kl + (\beta/\sigma)\cos kl}, \quad . \quad . \quad . \quad (1)$$

for the impedance, Z_1, at one end of a tube of length l † and area of cross-section σ in terms of the impedance, Z_2, at the other end, and β the product of the elasticity of the gas into k ($= 2\pi n/a$ with the usual notation);

$$\text{and (B)} \qquad Z_1' = -\beta k/c, \quad . \quad , \quad . \quad . \quad . \quad . \quad (2)$$

for the impedance of an orifice of conductance c.

* Communicated by the Author.

† Throughout the paper the "length" of a pipe is to be considered as its geometrical length together with any end correction or corrections.

By the use of these formulæ it is easy to write down an expression for the resonant frequencies of the system shown in fig. 1. For, by (1), the impedance at Q due to the gas column of length l_2 being

$$\frac{Z_P-(\beta/\sigma)\tan kl_2}{1+Z_P(\sigma/\beta)\tan kl_2},$$

$$Z_Q=-\beta k/c+\frac{Z_P-(\beta/\sigma)\tan kl_2}{1+Z_P(\sigma/\beta)\tan kl_2}, \quad . \quad . \quad . \quad (3)$$

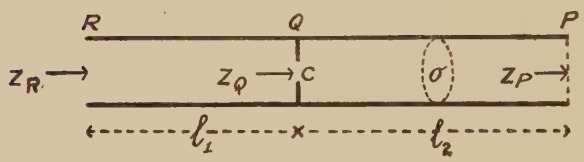
and similarly

$$Z_R=\frac{Z_Q-(\beta/\sigma)\tan kl_1}{1+Z_Q(\sigma/\beta)\tan kl_1}.$$

If the system is resounding to a source at R (such that R is an antinode), then, neglecting friction, $Z_R=0$, or

$$Z_Q=(\beta/\sigma)\tan kl_1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Fig. 1.



If the tube be open at P, $Z_P=0$, and from (3) and (4)

$$\tan kl_1+\tan kl_2=-k\sigma/c, \quad . \quad . \quad . \quad . \quad (5)$$

or, if it be closed, $Z_P=\infty$, and

$$\tan kl_1-\cot kl_2=-k\sigma/c. \quad . \quad . \quad . \quad . \quad (6)$$

Formulæ (5) and (6) lend themselves to a graphical treatment (due to Paris[5]) which has been used by the present author to demonstrate the effect of side-holes on the frequency of a wind instrument[2]. Put

$$f=\frac{l_1}{l_1+l_2}=\frac{l_1}{L},$$

so that

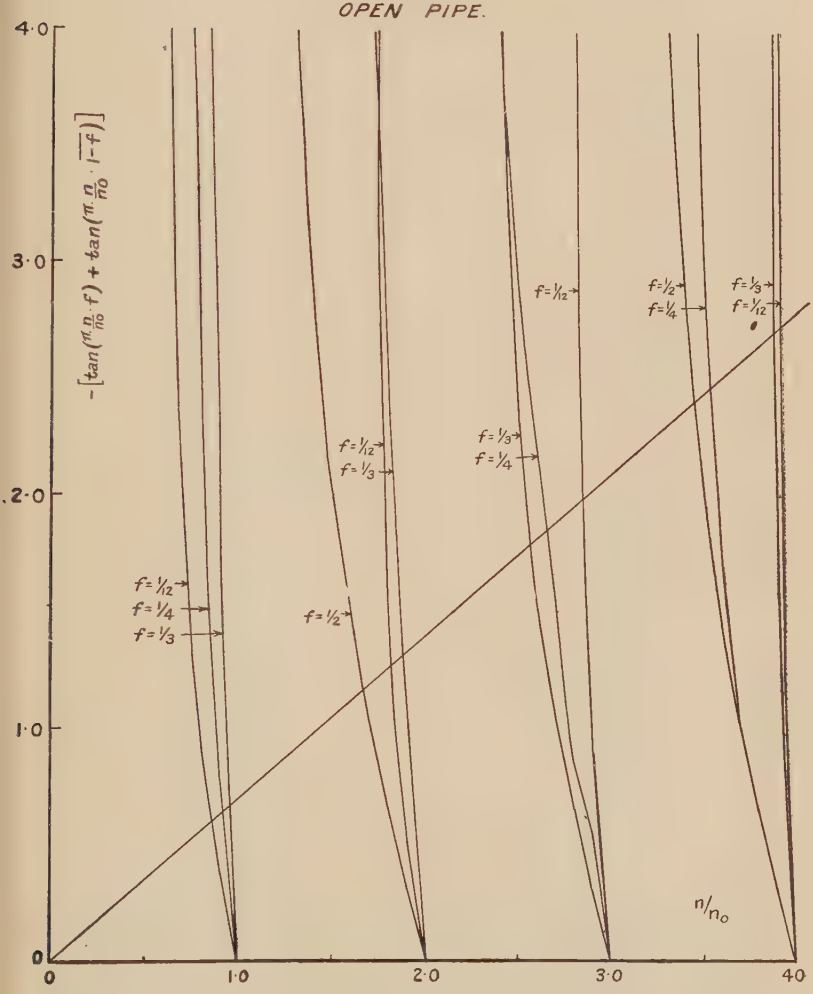
$$(1-f)=\frac{l_2}{L},$$

and let the frequency of the pipe when there is no constriction present be n_0. We then have $a=2Ln_0=2\pi n/k$ or

$k=(\pi/\mathrm{L})\,.\,(n/n_0)$ for an open tube, and $k=(\pi/2\mathrm{L})\,.\,(n/n_0)$ for a closed tube. Equations (5) and (6) then become

$$-\left(\tan\pi\,.\,\frac{n}{n_0}\,.f+\tan\pi\,.\,\frac{n}{n_0}\,.\,\overline{1-f}\right)=\frac{\pi}{\mathrm{L}}\,.\,\frac{n}{n_0}\,.\,\frac{\sigma}{c},\quad\ldots\quad(5.1)$$

Fig. 2.

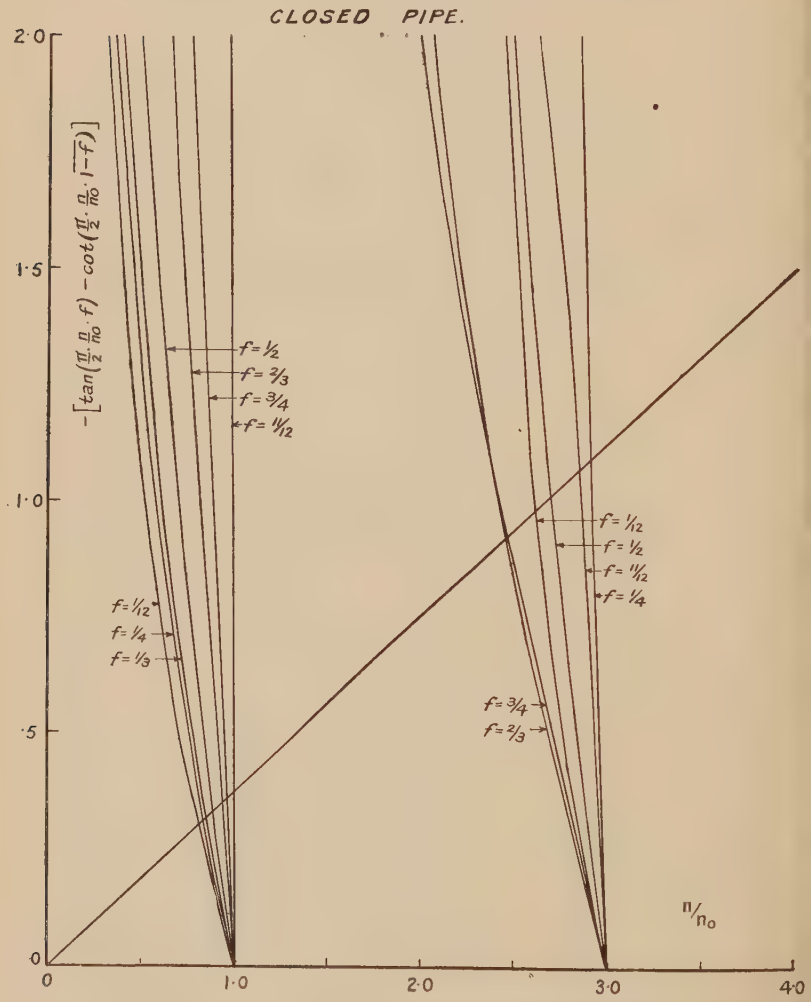


and

$$-\left(\tan\frac{\pi}{2}\,.\,\frac{n}{n_0}\,.f-\cot\frac{\pi}{2}\,.\,\frac{n}{n_0}\,.\,\overline{1-f}\right)=\frac{\pi}{2\mathrm{L}}\,.\,\frac{n}{n}\,.\,\frac{\sigma}{c}.\quad\ldots\quad(6.1)$$

Plotting values of the left-hand sides of these equations against n/n_0 for various values of f gives rise to the curves of figs. 2 and 3 respectively. Plotting values of the right-hand sides

Fig. 3.



for given constrictions in pipes of known length and cross-section against n/n_0 gives rise to a series of straight lines on the same figures. It is to be noted that the curves, as distinct from the straight lines, will serve for any pipe for which the value of n_0 is known.

It is of *theoretical* interest to note from figs. 2 and 3 the values of the frequencies indicated for the cases in which c is very large (equivalent to removing the constriction) and very small (equivalent to blocking the tube), and to compare them with the corresponding "stationary wave diagrams" of the pipe. The values of n/n_0 in these instances are set out in Table I.

TABLE I.

	Open Tube.		Closed Tube.	
	$c=0$.	$c=\infty$.	$c=0$.	$c=\infty$.
f.	(Gradient * $=\infty$.)	(Gradient=0.)	(Gradient=∞.)	(Gradient=0.)
1/12.....	6/11	1	0	1
	18/11	2	24/11	3
	30/11	3		
	42/11	4		
1/4	2/3	1	0	1
	6/3	3	8/3	3
	10/3	4		
1/3	3/4	1	0	1
	6/4	2	3	
	9/4	3		
	15/4	4		
1/2	1	2	0	1
	3	4	2	3
2/3	3/4	1	0	1
	6/4	2	3/2	3
	9/4	3		
	15/4	4		
3/4	2/3	1	0	1
	6/3	3	4/3	3
	10/3	4		
11/12.....	6/11	1	0	1
	18/11	2	12/11	3
	30/11	3		
	42/11	4		

* The gradient referred to is that of the straight lines.

It is to be noted that:

1. In the columns for $c = \infty$ are to be found the overtones proper to the tube without any constriction, except in those instances in which a node would occur at the position of the constriction were it present when these overtones appear under the column $c = 0$.

2. The values under $c = 0$ are, as might be expected from energy considerations, associated with that part of the tube

which, if sounded by itself, would yield the lower frequencies (*i. e.*, in general, the longer part).

3. The values $n/n_0 = 0$ for $c = 0$ in a closed tube may be interpreted as the note associated with a tube of infinite length.

The value of the "conductance" of a hole in a disk perpendicular to the axis of a tube has not yet received theoretical treatment. In consequence c was determined experimentally by plotting the observed values of n/n_0 for various values of f on figs. 2 and 3, drawing the best straight line from the origin through the points so obtained, and measuring the gradient of this line. This is in some measure tantamount to assuming the validity of formulæ which it is the object of experiment to test. The justification for this procedure is that the experimental values of the frequencies do not differ greatly from those calculated assuming throughout the same value of c for the constriction used. It is, however, possible that c may vary slightly with the frequency and the proximity of the constriction to the mouth of the tube, and this may account to a certain extent for the existing discrepancies.

3. *Experimental.*

The "boot" of an open cylindrical flue organ pipe designed to emit E (= 320 cycles/sec.) was severed from the rest of the pipe and fitted with a collar to carry one of three brass tubes of length 100, 60, and 35 cm. respectively, and of internal diameter approximately equal to that of the original pipe (4·7 cm.). The constriction used was of brass of thickness 0·2 cm., having a circular aperture, of diameter ·84 cm., coaxial with the pipe.

As a preliminary it was necessary to determine the end correction of the pipe and its variation (if any) with frequency. To this end the 100 cm. pipe was furnished with a closely fitting piston, and the pipe lengths corresponding to the normal tones or harmonics of a series of tuning-forks were determined. If L, n, and C are the length, frequency, and mouth correction (supposed constant) of the pipe, then

$$\mathrm{L} = (a/4).1/n - \mathrm{C}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

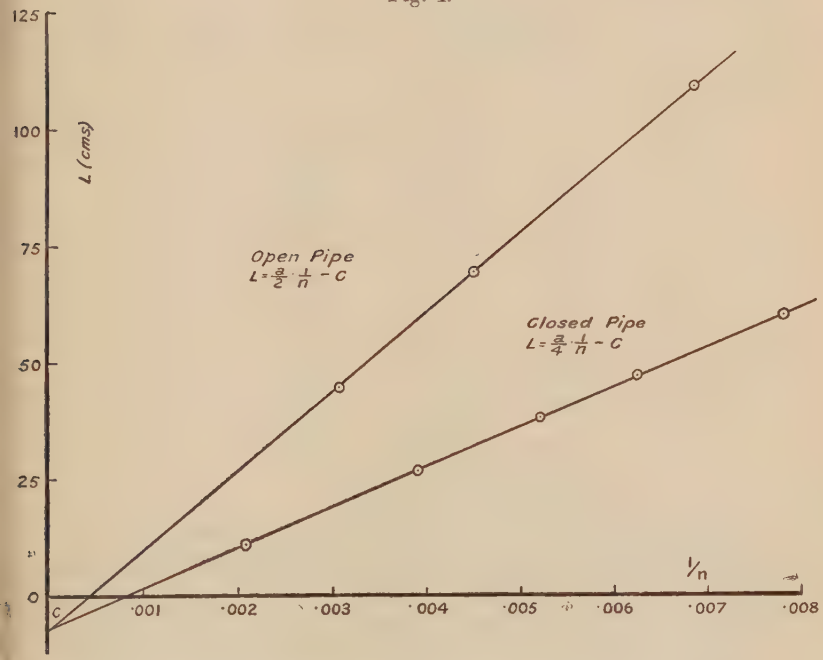
where a is the velocity of sound. The frequencies obtained by using the mouthpiece in conjunction with the three unstopped brass pipes were also determined (by a sonometer), and in these instances

$$\mathrm{L} = (a/2).1/n - \mathrm{C}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where L is now to be interpreted as the geometrical length of the pipe increased by 0·58 times its radius. Plots of L against $1/n$ (fig. 4) are sensibly rectilinear, and indicate that the mouth correction 6·9 cm. (= 2·9 times tube radius *), may, for the purposes of this work, be taken as the same whether the pipe be closed or open and also as independent of the frequency †.

In making these measurements care was taken to ensure that the pipe was neither over- nor under-blown for the particular note under test, and it was found that, with the

Fig. 4.



* This value is in agreement with that obtained for a flageolet [2].

† If it be assumed that the mouth correction is a *linear* function of L (= C+bL, say), then

$$L = \frac{a}{4n(1+b)} - \frac{C}{(1+b)}$$

for a closed pipe, and a straight line is, for this particular function, still obtained by plotting L against $1/n$. The measured gradients of the lines of fig. 4 yielded values of the velocity of sound equal to 342 and 343 metres per sec. for an open and closed pipe respectively, so that, having regard to the temperature of the air in the pipes (19° C), it appears safe to assume that $b = 0$.

aid of a water manometer, consistent results could be obtained after a little practice.

Another and similar pipe was calibrated in the same manner as the experimental pipe and a satisfactory value of the velocity of sound was deduced from the gradient of a plot of L against $1/n$. This pipe was then ready for use to determine the frequencies of the notes emitted by the experimental pipe with a constriction in position: these comparisons were facilitated by the counting of beats.

In Table II. are shown the frequencies (expressed as n/n_0 in those instances for which $n/n_0 < 3$) obtained experimentally,

TABLE II.

	Open Pipe.		Closed Pipe.	
f.	Calc.	Obs.	Calc.	Obs.
1/12.....	·85	*	·82	*
	1·82		2·63	
1/4	·92	·93	·84	·81
	2·64	2·69	2·93	2·93
1/3	·95	·97	·86	·84
	1·88	1·90		
1/2	1·67	1·66	·89	·87
			2·71	2·75
2/3	·95	·94	·93	·94
	1·88	1·96	2·46	2·56
3/4	·92	·91	·97	·97
	2·64	2·63	2·47	2·58
11/12.....	·85	·83	1·00	·99
	1·82	1·83	2·86	2·90

* Due to a slight tapering of the pipe "boot" it was not possible to place the constriction in the place corresponding to $f = 1/12$.

In some instances other notes, normal overtones of the pipe without a constriction and tones to be expected if the diaphragm formed a node, were elicited by appropriate blowing.

and from figs. 2 and 3 for the fundamental and first overtone of the 60 cm. brass pipe †. The conductance of the hole in the diaphragm, determined in the manner described above, was 1·02 cm. ‡ With the kind assistance of Mr. W. Lethersich, B.Sc., the results were verified approximately by means of a sonometer.

† The corrected length of (*a*) the open pipe (including end corrections and "boot" length) was 76·5 cm., and (*b*) the pipe when closed by a 3·2 cm. thick brass stop was 71·9 cm.

‡ With a smaller wave-length than any used in these experiments a smaller value for the conductance of a hole of this size has been determined.

It is considered that the agreement between the "theoretical" and actual results is sufficient to substantiate the correctness of the theory proposed.

It is of interest to note that M. de Mons places his constriction near the mouth of a stopped pipe and also decreases the conductance of the orifice by making it in the form of a short tube instead of a simple hole in a diaphragm: the present theory shows that both these procedures tend to minimize the frequency of the note.

I have pleasure in tendering my thanks to Professor Lees for placing the resources of his laboratory at my disposal for the purposes of this work, and to Messrs. Spurden Rutt & Co., of Leytonstone, for supplying the pipe "boots."

References.

(1) Irons, Phil. Mag. v. p. 580 (1928).
(2) Irons, Phil. Mag. x. p. 16 (1930).
(3) Irons, Phil. Mag. ix. p. 346 (1930).
(4) Webster, Proc. Nat. Acad. Sci. v. p. 275 (1919).
(5) Paris, Phil. Mag. xlviii. p. 769 (1924).
(6) Irons, Phil. Mag. vii. p. 873 (1929).

LXXXVII. *The Orientation of Rolled Aluminium.* By J. Thewlis, *M.Sc., Physics Department, National Physical Laboratory, Teddington, Middlesex* *.

[Plates VIII.–X.]

THE present investigation on the rolling of aluminium is divided into two parts. The first part deals with the flat-rolling of aluminium, which has been the subject of divergent opinion, and the second part deals with the square-rolling of aluminium, which has not previously been investigated. The latter refers to a specimen which has been rolled in such a way as to preserve a square cross-section throughout the rolling process.

1. *The Orientation of Flat-rolled Aluminium.*

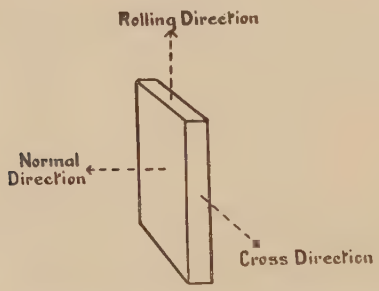
The results of previous investigations on the flat-rolling of aluminium, with the exception of those of Göler and Sachs†,

* Communicated by Dr. G. W. C. Kaye, Superintendent.
† v. Göler and G. Sachs, *Z. f. Phys.* xli. p. 873 (1927).

may be divided into two groups. In one * the direction of rolling is given as a [111] direction and the normal direction as a [211], and in the other † the rolling direction is given as a [211] direction and the normal direction as a $[01\bar{1}]$. Göler and Sachs find that the rolling direction is in the neighbourhood of a [335] direction and the normal direction is in that of a $[\bar{5}31]$.

In the present investigation two series of photographs were taken of a specimen of flat-rolled aluminium, the X-ray beam making various angles with the normal direction (fig. 1). The rolling direction was vertical in one series and the cross direction was vertical in the other. The X-rays (from a molybdenum target) passed straight through the specimen, a photographic plate being set up at

Fig. 1.



right angles to the beam. The photographs consisted of spots lying on powder rings. The intensity and position of each spot varied from photograph to photograph. Each plane reflected to some extent in almost every position of the specimen, indicating a somewhat incomplete selective orientation.

Fig. 2 (Pl. VIII.) and fig. 3 (Pl. IX.) illustrate the kind of photograph obtained.

If it could be decided in what position of the specimen a given spot had its maximum intensity on the photograph, it would be easy to determine the predominating orientation. Unfortunately the spots appear to be of the same intensity in some cases for a range of 20° or 30° in the

* F. Wever, *Z. f. Phys.* xxviii. p. 69 (1924); Owen and Preston, Proc. Phys. Soc. xxxviii. p. 132 (1926).

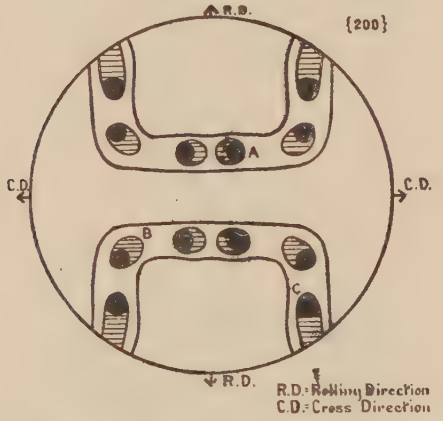
† Uspenski and Konobejewski, *Z. f. Phys.* xvi. p. 215 (1923); Mark and Weissenberg, *Z. f. Phys.* xiv. p. 328 (1923).

position of the specimen. This is due partly to the difficulty of comparing relative intensities on separate negatives, and partly to the incompleteness of the selective orientation. It is therefore necessary to devise a method for finding the mean position of maximum reflexion for each plane.

The method adopted will be illustrated by considering its application to the {200} planes, these being the easiest to discuss.

Corresponding to each spot on any photograph there must be a certain "reflexion plane" of the specimen in a definite position. This position was determined for each of the {200} spots in both series of photographs, and the

Fig. 4.



poles of the various "reflexion planes" were plotted on a stereographic projection of the specimen. The projection is shown in fig. 4, and maps out the areas in which the {200} poles lie.

The shaded portions correspond to intense spots on the photographs, and therefore to concentrations of {200} poles. The problem is to find positions of maximum preference in these portions.

Consider the area A (fig. 4). One of the required poles is within this area, and there must be two other corresponding poles 90° away from it and from one another. The only areas in which these can lie are those marked B and C. Owing to the difficulty mentioned above of comparing relative intensities on separate negatives the areas

A, B, and C are larger than they would have been if their extent had been entirely due to incomplete selective orientation. These additions to the true areas may be detected by their having no corresponding areas 90° away from them. Thus it is obvious that some of the points in

Fig. 5.

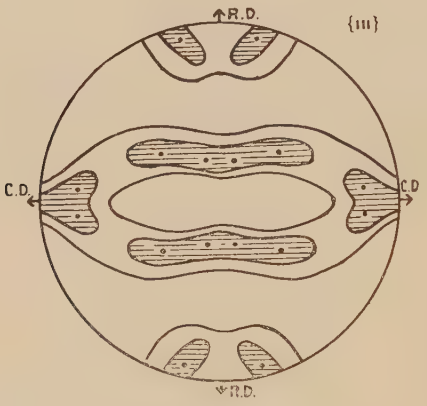
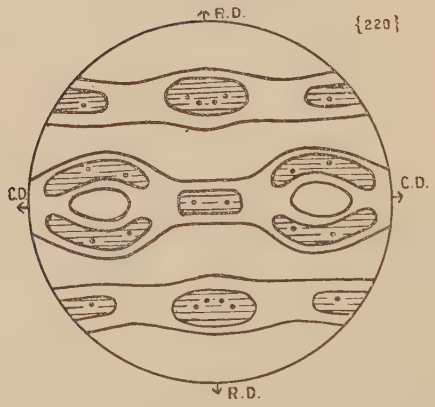


Fig. 6.



B have no corresponding points 90° away from them in A, so that the area of B which need be considered can be reduced. Similarly, by taking the areas in pairs portions of A and C can be eliminated, enabling B to be cut down still more. The process can be continued until all portions of the areas not having corresponding parts 90° away from them have been eliminated. The areas left as a result of

this process are represented by black areas in the diagrams. A maximum has been taken to be at the middle of each black area. It will be noticed that there are twelve such areas; hence four sets of crystals occur.

Figs. 5 and 6 show corresponding diagrams for the {111} and {220} poles respectively. The final black areas have been omitted for simplicity, the positions of maximum preference being indicated by dots. Fig. 7 shows the maxima of the poles of all three types of plane, for one set of crystals, plotted on the same diagram. From the angles which the poles of fig. 7 make with the rolling direction and the normal direction, the direction cosines of these two directions with respect to the cube axes [100], [010], and [001] may be calculated. These (expressed as angles) are shown in Table I.

Fig. 7.

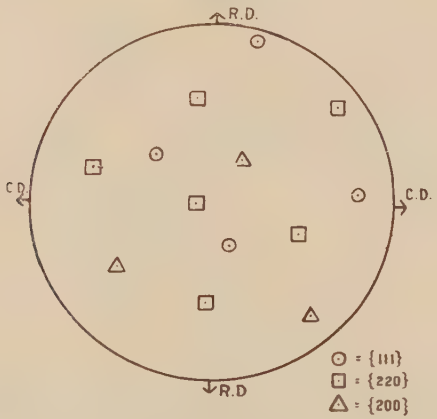


TABLE I.

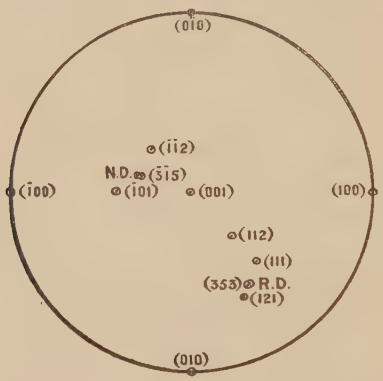
Angle between:	[100].		[010].		[001].	
and:	R.D.	N.D.	R.D.	N.D.	R.D.	N.D.
Calc. from {111} planes.	62°·9	123°·0	40°·0	98°·9	62°·5	35°·6
Calc. from {200} planes.	59°·0	118°·0	44°·5	103°·0	62°·0	31°·0
Calc. from {220} planes.	64°·0	127°·0	38°·6	96°·2	63°·8	38°·2
Mean	62°·0	122°·7	41°·0	99°·4	62°·8	34°·9
Direction cosines ..	0·470	—0·540	0·755	—0·163	0·457	0·820

The direction cosines of the rolling direction are seen to be very nearly in the ratio 3 : 5 : 3, and those of the normal direction to be very nearly in the ratio $-3 : -1 : 5$.

From Table I. the positions of the rolling direction and normal direction were plotted on a stereographic projection of the crystal, the plane of projection being the (001). This is shown in fig. 8.

It can be seen that the rolling and normal directions are not far from [353] and [$\bar{3}\bar{1}5$] directions respectively. It must be remembered, however, that the rolling and normal directions occur within appreciable ranges, and must not be associated with definite directions, crystallographic or non-crystallographic.

Fig. 8.



It has already been mentioned that investigations previous to that of Göler and Sachs gave either a [111] or a [211] as the direction of rolling, with a [211] and a [01$\bar{1}$] as normal direction respectively. Göler and Sachs * suggest that these two orientations might exist together, and by overlapping give a result similar to that found. In this case one would expect the areas in figs. 4, 5, and 6, corresponding to strong reflexions, to contain the positions of poles which would occur if the two orientations in question existed. Accordingly the positions of these poles were marked on figs. 4, 5, and 6, and collected on one diagram (fig. 9). The points refer to the orientation with a [111] as direction of rolling, and the crosses refer to that with a [211] as rolling direction.

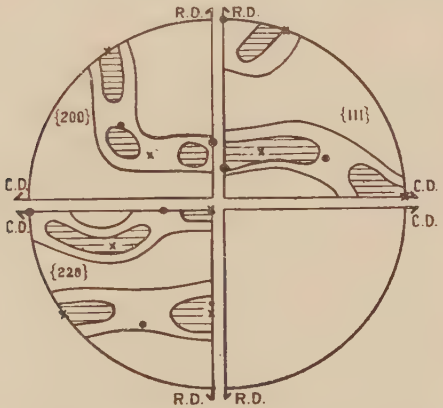
* *Loc. cit.*; *Zeit. f. Phys.* lvi. p. 477 (1929).

It will be seen that the areas of strong reflexion lie in general between the poles. If the result is to be explained by the overlapping of the two orientations, then it must be assumed that these orientations are very indefinite. This is equivalent to the statement that the rolling direction and normal direction can each occur within a big range. There seems to be no reason for assuming the existence of these two orientations as such.

2. *The Orientation of Square-rolled Aluminium.*

The specimen (which before square-rolling had been carefully annealed, and on examination by X-rays had

Fig. 9.



given no sign of selective orientation) was mounted on a goniometer. As before, a fine beam of X-rays from a molybdenum target was passed straight through the specimen, a photographic plate being set up at right angles to the beam.

The rod was mounted with the direction of rolling vertical, and a series of photographs was taken in which the X-ray beam made different angles with a face of the specimen. In each case the photograph was the same as that obtained when the specimen was rotated about the rolling direction. This photograph is reproduced in fig. 10 (Pl. X.). Such a result suggested that the crystals of which the rod was composed had one crystallographic direction in common, but were otherwise randomly oriented, the common direction being parallel to the direction of rolling.

This was confirmed by taking photographs in which the direction of rolling was horizontal, the specimen being rotated about a vertical axis. The same picture was obtained whatever the inclination of the specimen faces to the vertical. Such a photograph is reproduced in fig. 11 (Pl. X.).

The direction of rolling may be determined from fig. 10 (Pl. X.). It is found to coincide with a [111] direction. There is also evidence of the presence of a second orientation with a [100] direction in the direction of rolling. Table II. gives the angles between the normals to the various planes and the rolling direction as observed and as calculated for a [111] as rolling direction and a [100] as rolling direction.

TABLE II.

Plane (h. k. l.)	Angle between normal to plane and rolling direction.		
	Observed.	Calculated.	
		[111] as R.D.	[100] as R.D.
111	54°·9	—	54°·8
	70°·7	70°·6	—
200	55°·1	54°·8	90°·0
220	35°·4	35°·3	45°·0
	90°·0	90°·0	90°·0

Thus a square-rolled aluminium rod has the same orientation as a cold-drawn aluminium wire *.

In addition to the above, observations were made on the square-rolling of a rod which had previously been flat-rolled. The orientation of the rod before square-rolling was found to be the same as that described in Part 1. After square-rolling it was found that the direction of rolling was a [111] direction, the normal to one pair of faces a [110], and the normal to the other pair a [112]. It is hoped to pursue this investigation at a subsequent date.

The author is indebted to Dr. G. Shearer for his continued interest and helpful criticism, and to Prof. G. I. Taylor, who kindly furnished the specimens.

* Ettisch, Polanyi, and Weissenberg, *Z. f. Phys. Chem.* xcix. p. 332 1921); Schmid and Wassermann, *Z. f. Phys.* xlii. p. 779 (1927).

Summary.

The orientation of a flat-rolled aluminium sheet has been determined by a graphical method. It has been found to be such that no definite direction in the crystal can be identified with the rolling direction or the normal direction. The mean position of the rolling direction is near a [353] direction, and that of the normal direction is near a [$\bar{3}$15] direction. Four sets of crystals occur, the plane of the rolling and normal directions and the plane of the cross and normal directions behaving like mirror planes. This result confirms that of Göler and Sachs.

The effect of the square-rolling of aluminium has been investigated. It has been found that a square-rolled rod, if rolled from an aluminium rod possessing random orientation, has the same orientation as a cold-drawn aluminium wire.

It has been found that the square-rolling of an aluminium rod which has previously been flat-rolled produces an orientation in which the rolling direction is a [111] direction, the normal to one pair of faces a [1$\bar{1}$0] direction, and the normal to the other pair a [11$\bar{2}$] direction.

LXXXVIII. *Notices respecting New Books.*

The Zeta-Function of Riemann. By E. C. Titchmarsh, M.A. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 26.) [Pp. vii + 104.] (Cambridge: At the University Press, 1930. Price 6*s.* 6*d.* net.)

IN this tract an account is given of the theory of the zeta-function, without regard to its applications in the theory of numbers, which is to form the subject of a companion tract. The better-known properties of the function are briefly sketched in an introductory chapter. The first chapter is devoted mainly to inequality theorems, with a view to the study of the asymptotic behaviour of $\zeta(\sigma+it)$, as $t\to\infty$ for given values of σ. It is shown that there is a critical strip within which the problem of the order of the function is unsolved. The second chapter deals with mean value theorems, and it is shown that the problem of the average order or mean value of the function in the critical strip is much easier. The important problem of the distribution of the zeros of the function is discussed in the third chapter,

along with the famous "Riemann hypothesis," that all the complex zeros of $\zeta(\sigma+it)$ lie on the line $\sigma=\frac{1}{2}$. This is followed in the next chapter by the wider problem of the general distribution of the values of the zeta-function; this chapter is in part adapted from a manuscript prepared by Professors Bohr and Littlewood. In Chapter V. some of the consequences of the improved Riemann hypothesis are investigated; if the hypothesis is false, it might be hoped that in this way a contradiction would sooner or later be met with, but so far the theory based on the hypothesis has proved perfectly coherent. The hypothesis enables more precise results to be deduced about the vertical distribution of the zeros and about order problems. In a final chapter some consequences of Lindelöf's hypothesis are considered. This hypothesis is more restricted than Riemann's, and does not necessarily involve the truth of the latter, although, on the other hand, if Riemann's hypothesis is true, Lindelöf's hypothesis is also true.

The tract provides an excellent summary of zeta-function theory, clearly though concisely expressed. The most recent investigations are incorporated. An extensive bibliography is added at the end, to which references are given throughout the text.

A Comprehensive Treatise on Inorganic and Theoretical Chemistry. By J. W. Mellor, D.Sc., F.R.S. Vol. x. S, Se. [Pp. x + 958, with 66 figures.] (London: Longmans, Green & Co., 1930. Price £3 3*s*. net.)

The publication of Dr. Mellor's "magnum opus" is proceeding at the rate of one volume per year. Three further volumes have yet to be published. The new volume is devoted to the allied elements, sulphur and selenium: 692 pages are given to sulphur and 266 to selenium. Both of these elements have physical properties of considerable interest, and emphasise the advantages of Dr. Mellor's arrangement, which deals in some detail with both physical and chemical problems. The sections which deal with the history, occurrence, methods of extraction, and use contain much of general interest. Under compounds of sulphur, sulphuric acid naturally occupies a prominent place and receives very full treatment.

With the progress of the publication of this great work, one becomes more and more amazed at the industry of the author, at the manner in which the standard of the initial volumes has been maintained, and at the thoroughness with which not only chemical but also physical literature has been combed almost up to the date of publication. The work is remarkably complete, and fullest references to original papers are given. There are doubtless some omissions, but

a careful search through the present volume has failed to reveal any of importance. Each volume contains a detailed subject-index. No chemical library can afford to be without Dr. Mellor's 'Comprehensive Treatise.'

An Outline of Wave Mechanics. By N. F. Mott, Lecturer in Theoretical Physics, The University, Manchester. (Cambridge University Press, 1930. Price 8*s.* 6*d.*)

This interesting book is based on a course of lectures given by Mr. Mott in the University of Manchester in 1929–30.

Mr. Mott's aim has been to give an account of the New Quantum Theory which should be intelligible to the advanced student of experimental physics and to the research worker.

The author presupposes an acquaintance with Fourier analysis and with the elementary principles of wave-motion, but so excellent is the exposition that the intelligent student—to whom it is addressed—should have no difficulty in following any mathematical argument which is introduced.

The reviewer would like specially to recommend this volume not only to students of physics, but also to the applied mathematicians.

A History of Science. By Wm. Cecil Dampier-Whetham, M.A., F.R.S., Fellow and sometime Senior Tutor of Trinity College, Cambridge; Fellow of Winchester College. (Cambridge University Press, 1929. Price 18*s.*)

In this interesting volume Mr. Dampier-Whetham makes an attempt to give a complete outline of the development of scientific thought. The general plan is based on a book entitled 'Science and the Human Mind,' published by Mr. Dampier-Whetham as long ago as 1912. The present volume also contains many of the ideas in the chapter on the "Scientific Age" of the Cambridge Modern History and the article "Science" in the eleventh edition of the 'Encyclopædia Britannica.'

It is of great value to the general reader to have this important material welded together in one compact volume.

Mr. Dampier-Whetham has also, of course, published a number of other essays on the subject, including an interesting chapter in Harmsworth's Universal History.

The volume is notable for the large number of references which would enable the student of the subject to read more and more deeply in this important branch of history.

At the present time, when fundamental advances are being made in so many different directions, there is a danger that the history of science may not be sufficiently studied by the young student. This fascinating and interesting work should certainly do something to counteract this tendency.

Optics. By B. K. Johnson, F.R.M.S., Demonstrator in the Technical Optics Dept. of the Imperial College of Science and Technology, Lecturer in the Applied Optics Dept. of the Northampton Polytechnic Institute (London). (London : Edward Arnold & Co., 1930. Price 8*s.* 6*d.*)

This interesting and useful book serves as a convenient reference book for a course of lectures and experiments in optics.

There appears to be a modern tendency for mathematical students to study optics without sufficient experimental background. To the present reviewer it would seem an admirable plan if the ordinary theoretical reading of the mathematical student were supplemented by a course of experiments. In such a scheme the present book would prove of the greatest value.

This book, however, has no doubt as its primary purpose the education of the students of physics, and the author has certainly succeeded admirably if this is his aim.

It is a feature of the book that there is a detailed description of all the apparatus required, so it should prove exceedingly handy for the assistant who is responsible prior to the lecture for seeing that everything is ready to hand.

A word must be added about the diagrams, which seem specially successful.

Traité de Polarimétrie. Par Prof. Georges Bruhat. [Pp. xvi + with 250 figures.] (Paris : Éditions de la *Revue d'Optique theorique et instrumentale,* 1930. Price 65 frs.)

Professor Bruhat has defined, in his introduction, the subject of this book as the study of the phenomena of rotatory polarization and of the apparatus used for their measurement. With subject-matter so defined, the book naturally falls into two sections, one dealing with the instrumental equipment and the methods of making the measurements, and the other concerned with the observed phenomena and their applications.

The phenomena associated with the radiation of the plane of polarization are of interest to the physicist, to the laboratory chemist, and to the industrial chemist in certain industries. The author has had all three classes in mind in writing this book, and has aimed at producing a volume which shall be self-contained, in the sense that a chemist, for instance, using the book will not need to turn to another book to find the meaning of a physical term. The physical principles involved in polarization and double refraction are therefore developed at some length. The book suffers to some extent from over elaboration in this respect, though this is an error on the right side. The instruments described and their use are explained

with great care, and all who use such apparatus will undoubtedly derive much benefit from the study of this book. The author remarks that chemists will only use apparatus which is supplied in perfect order by the maker, but that he has deemed it necessary that they should know the defects and good points of different apparatus and why the most expensive apparatus is effectively the best.

In the second part the chief phenomena of rotatory polarization are discussed and theories of stereo-chemistry are considered. A considerable space is naturally devoted, both as illustrating the methods and also on account of its intrinsic importance, to the determination of sugar contents. The two final chapters deal with rotatory effects of crystals and magnetic rotatory polarization.

An extensive bibliography is given which will be found of great value, and a good index is provided. The bibliography emphasises the large amount of work which has been done in this subject and the rapid developments in recent years. The volume is well provided with illustrations of instruments and with numerous diagrams. The paper on which it is printed is of poor quality.

La Topographie sans Topographes. Traité de Photogrammétrie. Par F. Ollivier. [Pp. xviii + 301, with 158 figures.] (Paris : Éditions de la *Revue d'Optique théorique et instrumentale*, 1929. Price 42 fr.)

The application of photographic methods to topographical surveying is becoming of rapidly increasing importance, particularly in the case of mountainous regions, in view of the great saving in field-work which it permits and of the high order of accuracy which it can attain. This volume, by Captain Ollivier, of the French Army, gives an excellent account of the methods of observation, of the instruments used, and the manner in which the photographs are measured and reduced. It deals only with vertical photographs taken from points on the surface of the earth. Aerial surveying is being rapidly developed, and a further volume is promised in which the restriction to vertical photographs will be removed and the use of photographs taken under any conditions and in particular from aeroplanes will be discussed.

The volume begins with an interesting historical retrospect, from which the interesting fact emerges that as early as 1860 the French Colonel Laussedat had developed all the principles which were embodied at a later date in the Zeiss phototheodolite. The author then considers the general principles embodied in the observation of relief, and deals with binocular and stereoscopic vision and stereoscopic instruments. This provides a basis for the appreciation of the principles involved

in stereophotograms. These principles, both geometrical and instrumental, are carefully and clearly explained. A chapter is devoted to the Zeiss phototheodolite, the Zeiss stereocomparator and the stereoautograph. It is to be regretted that no account is given of the remarkable Wild phototheodolite, which the author would perhaps regard as not sufficiently tried out in practice. An interesting and important chapter deals with the question of the precision of the methods, and a comparison with the precision attainable by other methods of surveying is made; a detailed study of the various instrumental and other errors is given. This is followed by a further chapter dealing with the corrections required to take account of residual instrumental imperfections, with a view to obtaining the highest precision possible. The final section deals with observational procedure and methods of conducting surveys.

The volume provides not merely a very adequate account of the theory underlying the methods of photographic surveying, but deals with all the practical points involved from the time of securing the photographs to their final measurement and reduction. The author knows his subject thoroughly and has given an admirable exposition of it. The further volume, in which the limitation to vertical photographs is removed, will be awaited with much interest.

Cours d'Analyse, professé à l'Ecole Polytechnique. Tome II. Par J. Hadamand. [Pp. vi + 721, with 72 figures.] (Paris: Librairie Scientifique, Hermann et Cie, 1930. Price 140 frs.)

The general subjects treated in the second and concluding volume of Professor Hadamand's 'Cours d'Analyse' include potential theory, calculus of variations, the properties of analytic functions, differential equations, partial differential equations, the theory of probabilities, and the error law of Gauss.

Whilst adhering to a vigorous treatment throughout, the author has endeavoured to bring into the foreground various applications to neighbouring branches of science; dynamics, hydrodynamics, wave-theory, electrical theory, astronomy, and optical theory supply illustrations. The course is thus particularly well adapted for physical students. The treatment is not advanced, but is very comprehensive, and difficult points are carefully and clearly expounded. The general properties of harmonic functions are dealt with under the section concerned with potential theory; those of elliptic functions are dealt with under analytic functions. The solution of differential equations by methods of successive approximation is given prominence, in view of its practical importance.

LXXXIX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 368.]

June 11th, 1930.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S., President, in the Chair.

Slides illustrating frost stone-packing on the shores of Loch Lomond in relation to stone-polygons were exhibited by Prof. J. W. Gregory. He remarked that the stone-polygons of the Arctic soils had been described by Dr. C. S. Elton in the Quarterly Journal (vol. lxxxiii, 1927, pp. 163–93), and are well known from an extensive literature. According to the chief alternative theories, they are due either to pebbles being pushed outwards from a freezing centre which is left as a stoneless core of clay, or to repeated freezing reducing the central material to dust (Nansen's theory). When the President and himself landed in Spitsbergen with Sir Martin Conway's Expedition in 1896, their interest was at once aroused in these structures, which they regarded as the effect of repeated frost and thaw. So far as he knew, the structures are not definitely known from the Glacial drifts; he had seen embryonic developments in the soil in several places in Scotland, and they are well developed on the summit of the Merrick, the highest mountain in Scotland south of the Highlands. A development of this frost stone-packing occupies some acres on the eastern shore of Loch Lomond, on the Ross Promontory, on a beach formed of waste-fragments from an old slate-quarry. The slate fragments are vertical or steeply inclined; they are arranged in concentric series, often around boulders of the Lennoxian grits, and in places around tree-trunks. They occur in groups ranging from a few inches to a few feet in diameter. They are simple or multiple, regular or irregular in shape, and in some cases occur in bands composed of polycentric groups around oval depressions as much as 12 yards long; these long bands resemble the arcuate series in the Arctic.

This stone-packing is an example of solifluxion. The remarkable development beside Loch Lomond is clearly due to the repeated freezing of ground composed of flat slate-fragments, and irregularly charged with water; the absence of the mud core is due to the lack of fine-grained sediment; the vertical position of the stones indicates their packing and arrangement by horizontal pressure.

The slate-quarry was last worked 51 years ago, and the stone mosaic has probably been developed gradually during that interval.

June 25th, 1930.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S., President, in the Chair.

A lecture on the Tertiary Igneous Complex of Ardnamurchan was delivered by James Ernest Richey, M.C., B.A., F.R.S.E., F.G.S. The Lecturer said that Ardnamurchan is the

last of the well-known igneous complexes of Western Scotland to be studied in detail by the Geological Survey, and illustrates in a comparatively simple manner many of the remarkable features of Tertiary igneous activity in Britain. The mapping was carried out by Prof. E. B. Bailey, Mr. Simpson, and the Lecturer, and Dr. Herbert H. Thomas had been responsible for the petrology.

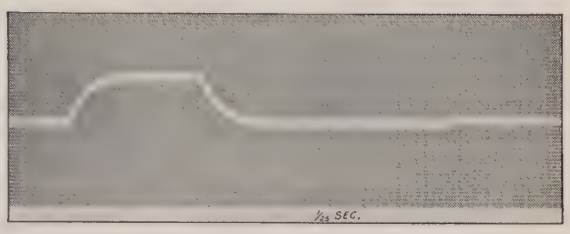
The district measures some 9 miles from east to west and 5 miles from north to south. It is chiefly noteworthy on account of its intrusive rocks, and only small outliers of the widespread Tertiary plateau basalt-lavas are preserved. The types of intrusion include volcanic vents piercing the basalt-lavas, and largely filled with acid and trachytic fragmental materials; minor intrusions, including cone-sheets, chiefly quartz-dolerite, and dykes; and plutonic masses, nearly all gabbro or dolerite, occurring mainly as ring-dykes. The above, excepting the dykes, are arranged in concentric series around three different centres, marking three foci of igneous activity which functioned successively.

Around Centre 1, situated on the east, there are extensive volcanic vents, a few major intrusions, and a set of massive cone-sheets. In relation to Centre 2, on the west, periods of cone-sheet and ring-dyke intrusion alternated, and there is only one small elongate vent. Country-rocks were first domed around this centre and then copiously injected by cone-sheets, including many of composite habit, with acid centres. These cone-sheets form an outer set around the succeeding ring-dykes. The intrusion of the ring-dykes was interrupted by a second, inner, suite of cone-sheets. Although only a single vent is developed, explosive brecciation of the walls of ring-dykes, or of entire ring-dykes themselves, is a recurring feature. The complex of Centre 3, situated midway between the two earlier centres, has obliterated large portions of those of earlier date. It consists of ring-dykes, with only a few cone-sheets cutting the outermost and earliest ring-dyke. The ring-dykes are, on the whole, successively younger as the centre is approached, and the more centrally situated include rock-types of rare occurrence in the British Tertiary province—biotite-bearing gabbros, tonalite, and quartz-monzonite. No vents are met with, and brecciation of the walls of ring-dykes is rarely seen. It is suggested that the three complexes are successively more deep-seated, due, presumably, to the growth of an overlying volcanic pile. The regular ring-patterns marked by the intrusions are, however, of more especial interest. They constitute evidence of the formation of annular or arcuate fissures that are considered here, as in Mull, to have resulted from localized stresses set up in the roof of an underlying magma-reservoir.

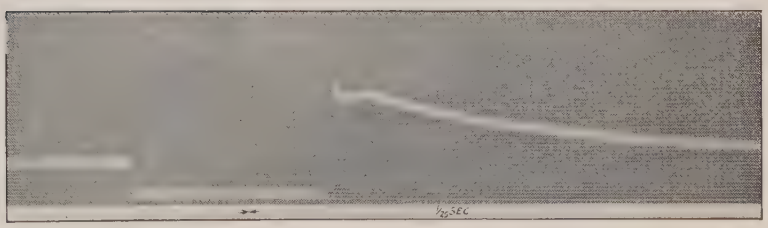
North-west dykes are numerous, and were intruded at various times during the period of localized activity, as well as after its close. The more westerly belong, in part at least, to the great Mull swarm.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

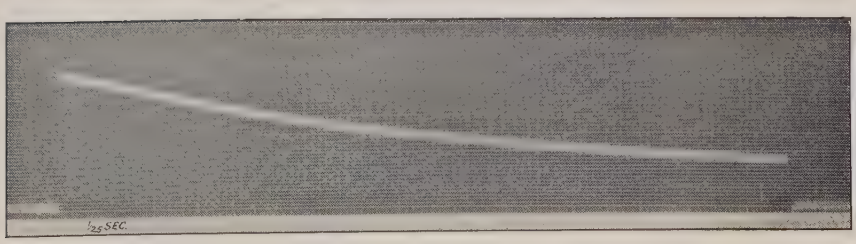
Fig. 3.



Curve 1.—Steady deflexion. Discharge.
Liquid paraffin "A," 52·5° C. Film thickness 0·4 mm.

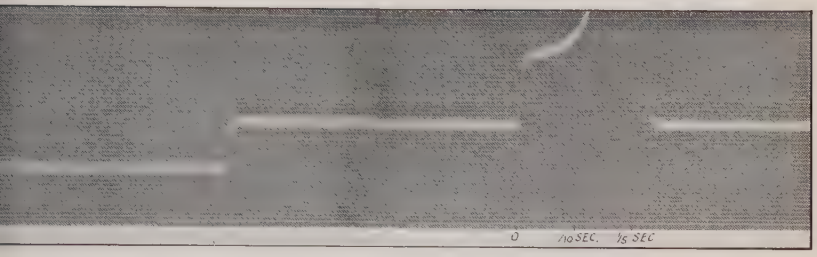


Curve 2.—Effect of 20·4 sec. short-circuit on liquid paraffin "B," 94° C.
(Steady deflexion before short-circuit shown on left.)

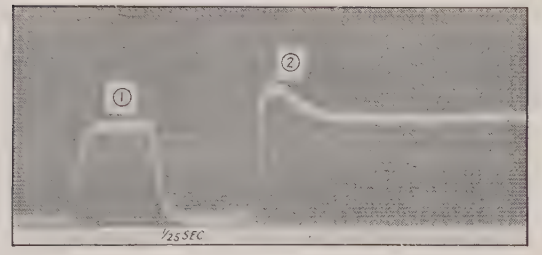


Curve 3.—Liquid paraffin "B," initial current, 350 v., 94° C.
(Time indication of moment of application omitted).

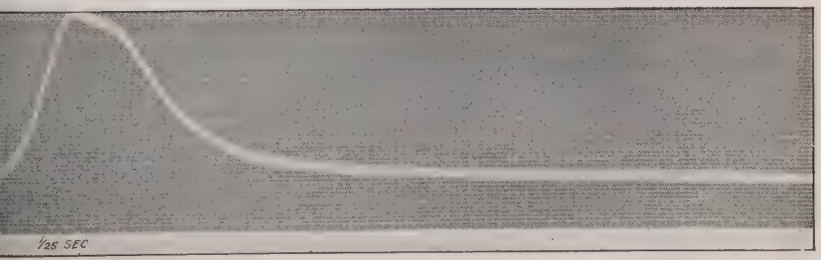
FIG. 8.



CURVE 1.—Deflexion before reversal. Reversal.
Liquid paraffin "A," 91° C. Film thickness 0·8 mm.

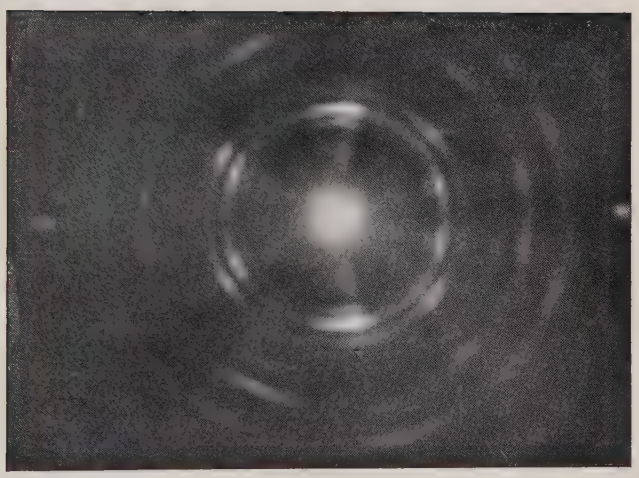


CURVE 2.—Increase in deflexion on double reversal: 1. Steady deflexion; 2. After second reversal. (Deflexion due to first reversal not shown.) Liquid Paraffin "A," 71° C. 0·4 mm. film.



CURVE 3.—Liquid paraffin "B" reversal current, 350 v., 80° C.

FIG. 2.



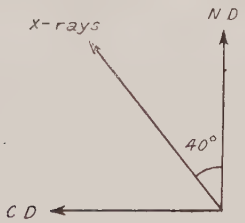
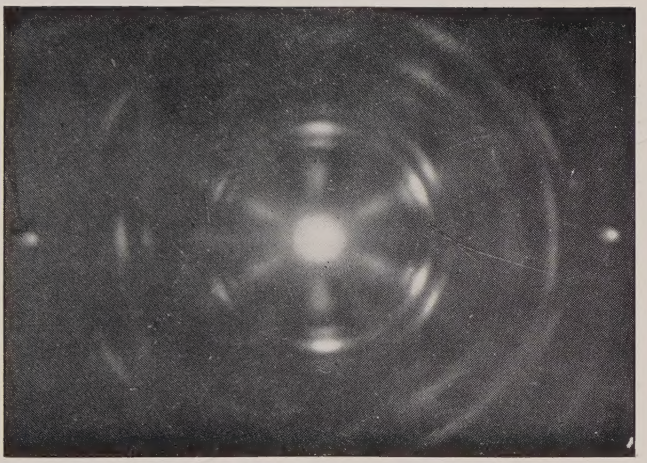


FIG. 3.



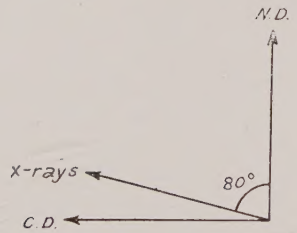


FIG. 10.

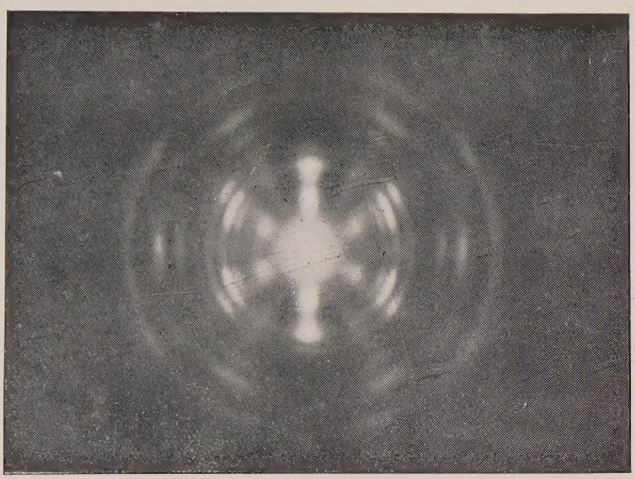


FIG. 11.